AN

ELEMENTARY TREATISE

OF

MECHANICAL PHILOSOPHY,

&c. &c.

AN

ELEMENTARY TREATISE

OF

MECHANICAL PHILOSOPHY,

WRITTEN FOR THE USE OF THE

UNDERGRADUATE STUDENTS

OF

THE UNIVERSITY OF DUBLIN.

BY

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PROVOST OF TRINITY COLLEGE, DUBLIN.

VOL. I.

SECOND EDITION.

DUBLIN:

PRINTED AT THE UNIVERSITY PRESS,

FOR MILLIKEN AND SON, 104, GRAFTON-STREET,

BOOKSELLERS TO THE UNIVERSITY.

M.DCCC.XXXV.

R. GRAISBERRY, Printer to the University.

TO

THE UNDERGRADUATE STUDENTS

OF

THE UNIVERSITY,

THE FOLLOWING TREATISE,

INTENDED FOR THEIR USE,

IS DEDICATED

BY THEIR VERY SINCERE FRIEND,

THE AUTHOR.

PREFACE.

It has ever been expected of an author, that in presenting himself to the notice of the public, he should state his claims to the attention he seems to solicit, and afford the means of judging how far it is likely to be rewarded. Anticipating these reasonable expectations, he endeavours to satisfy them by a declaration of the motives which may have led to the undertaking, or influenced his judgment respecting any peculiarity of plan or execution, by which his work may be distinguished. By affording this satisfaction to others, he also consults for his own interest or reputation; being well aware, that it is only by a due regard to the particular ends and objects of the writer, it can be ascertained whether his efforts are called for, his methods well chosen; or, even after the perusal, whether the work itself is fairly executed. This first duty to the public and

himself, the writer of the following Treatise shall endeavour to discharge as well as he is able.

Having been called to the chair of Natural Philosophy in the University of Dublin, he naturally felt it to be peculiarly his duty, in addition to his course of public lectures, to furnish a manual for the instruction of the students in his own department; the work long in their hands, consisting of lectures formerly delivered by Professor Helsham, being extremely imperfect, even as a system of Statics, to which branch it is almost exclusively devoted. In making this remark, it is not intended to cast any imputation on the memory of that writer. His volume conveys, in the most clear and familiar style, nearly all that was known on the subject when it first appeared. But whatever may be its peculiar merits, he who is aware of the vast improvements which have been made in the establishment and development of the principles of Mechanical Philosophy, needs only to be informed, that the work here spoken of appeared at the beginning of the eighteenth, and was probably written before the close of the seventeenth, century; and it is presumed that he will deem it unnecessary to inquire whether it contains, or not, a course of elementary instruction fitted for the present age.

Some attempt to supply the deficiency had indeed been made by the late Bishop Hamilton, author of a work on Conic Sections, fit to be placed beside the most finished productions of the ancient Geometers. But Natural Philosophy is chiefly of recent growth; and Elementary Treatises are the latest which arrive at perfection. During the progress of science, the task of its cultivators is that of exploring some region of vast extent: and it is not until their discoveries shall have swelled so as to approximate, that the whole can be combined into one continuous system. It is not, therefore, detracting from the merited reputation of that writer to assert, that the four supplementary lectures, first published about fifty years ago, are now only fitted to satisfy the curious as to the state of the science at that period.

Such was the acknowledged character of the manuals from which the students were to be instructed in this department of science: and the tutors, generally too much occupied in the discharge of their laborious duties to engage in a work for which some leisure was essentially requisite, contented themselves with supplying the deficiency of written treatises by the oral instruction of their

the author in this view, will also agree with him in supposing, that as the present attempt was not uncalled for, the only charge which can lie against him must relate to the quality of the work which he proposes to substitute for those already in use. For this indeed he has some apologies to offer, and a large share of indulgence to solicit. Such he hopes will not be withheld, when the purposes for which the work was more particularly intended are considered. Some explanation of these purposes may be requisite for the stranger to the plan of education adopted in the Dublin University. He is to be informed, that whilst it invites by honors to the highest attainments, and provides the most able assistance for those who may be so allured, it forbears from compelling attention to the subjects of its instructions beyond very moderate limits. The sound discretion manifested in this treatment of the younger members, is chiefly conspicuous in what relates to their mathematical studies. In a course of academic instruction, by which the youth of the country are to be qualified for the various professional duties of active life, and, therefore, necessarily embracing a considerable variety of subjects, it was not to be proposed that all should become profound mathe-

maticians. Neither could it be deemed expedient to interfere with the tastes of individuals, by which they may be directed into some other of the many walks of literature, perhaps equally useful, and certainly to many minds more inviting. In a society so diversified by tastes and objects, the number of those who do not enter on the higher branches of Mathematics must, at all times, be considerable. The qualifications of such, as well as of those of higher attainments among the students of the University, were necessarily to be attended to in a work proposed for their general use. Accordingly, in the following Treatise, the mathematical reasoning is sparingly resorted to, and as much as possible confined to the mere elements. Writing under these restrictions, and, in a certain degree, obliged to forego the aids peculiarly adapted to the science here treated of, the writer is apprehensive that he may have afforded to the mathematical reader some ground of complaint; yet it is to be hoped that he will not be dissatisfied with those sections which are more particularly intended for his perusal.

Such were the views of the author, and such the modifications of character which he has endeavoured to give to his performance. Should it be favourably

received by the University, as a work suited to the purposes for which it was intended, he shall feel encouraged to complete what is now done by the addition of a second volume: and for the judgment whereby it is to be decided, whether he is to proceed, or here to close his labours, he waits with some anxiety.

After explaining his views to the reader, the next duty of an author is to offer some acknowledgments for the aids he may have derived from his predecessors. Yet much in this way cannot be required from the writer of an Elementary Treatise, where there is little room for any pretension to originality. Most works of merit, wherein the writer could expect to find any thing connected with his subject, he has consulted; and by many of them he has profited. Among those to whom he is most largely indebted is Poinsot, whom he has followed by the adoption of his theory of couples, and his use of that doctrine for the establishment of the conditions of equilibrium. To this name must be added that of Poisson, an author not less remarkable for the depth of his views than for the elegance with which he unfolds them.

JANUARY, 1835.

ADVERTISEMENT

FOR THE SECOND EDITION.

The writer of the following pages had indulged the hope of completing this work on Mechanical Philosophy by the addition of another volume, before that a second edition of the first could be demanded. That hope has been frustrated by the engrossing duties of the situation to which he has been in the mean time called. The gentlemen for whose use it was chiefly intended, being aware of the nature and extent of his other engagements, will readily believe that he has done as much in this way as present circumstances would permit, by revising the work so far as it has been already accomplished, and which he now presents to them with several corrections, and such further improvements as were deemed compatible with the plan of the work.

CONTENTS.

INTRODUCTION.

The systems of matter which constitute this globe are ever liable to changes, with respect to their characteristic properties, their figures and motions; and it is to the observation of the influences, by which these changes are effected, that man is indebted for the foresight with which he anticipates the results of natural operations, and for the skill with which he avails himself of the various capabilities of matter. The processes which are constantly before his view, or those which are casually offered to his notice, would be the foundation of his first theories; and they would also furnish the light, for the direction of his earliest efforts, towards the improvement of his condition; for he would have only to subject the bodies at his disposal, to the same influences, in order to have the same results.

But the aim of the philosopher is not the servile imitation of nature; neither is the knowledge to which he aspires, limited to that of her most obvious proceedings. The actual operations of nature, are indeed the

subjects of his attentive observation; but he finds it requisite, by experimental investigation, greatly to enlarge the catalogue of these operations, when he attempts to discover the laws by which they are conducted, or the powers by which they are effected. Such is the scope of his endeavours; and having attained this point, he is enabled to anticipate the consequences of conditions, which nature had never presented; or of those, to which he is, himself, at liberty to subject the systems of matter placed at his disposal.

But the reward of his labours is not withheld, until the end of his researches is fully attained by him. Every step by which he advances leads him to more enlarged views of the power and wisdom of the Great Author and Contriver of all things, and of his provident concern for the well being of his Creation; and for every step, he is rewarded by an enlargement of his own power, over that world, which his Creator hath put under his dominion.

To trace his progress in this course, would be to write a history of the arts and sciences. The purpose of this introductory chapter, is to mark the subject treated of in those which follow: for the field of philosophical inquiry being of such vast extent, it is obviously requisite that it should be divided into separate compartments; and this division is the same with that of the phenomena to be considered.

Of these phenomena, the changes which occur in bodies as to rest and motion, constitute one important class; and the consideration of these changes, together

with that of their external causes, belongs to that department of Science, denominated Mechanical Philosophy.

We have spoken of these changes as being the effects of the actions of certain powers external to the bodies themselves; and without, for the present, deciding the question, whether there is in matter any innate tendency to rest or motion, to which such changes of condition might in certain cases be ascribed, it is obviously certain, that they are often the effects of external influences. We know that motion is frequently produced by an act of the will; and then as we are conscious of an effort which we designate by the name of force, so when the like effect is produced by any other external influence, it is natural to extend to that influence, the name with which we are already so familiar; and this we do, without intimating that there is any analogy whatever, between the powers exerted in the two cases. By the word force in mechanical philosophy, we denote, merely, the external cause of the change of condition as to rest or motion, without affecting to convey any notion of the more intimate nature of the energy so exerted: and happily, this acquaintance with the intimate nature of the force is not requisite for our present purpose; for in the investigation of the effects of certain powers, it is plainly unnecessary that we should know how they come to reside in the bodies by which they seem to be exerted; or even whether they actually do reside in those bodies. Thus when a bar of iron is disturbed on the approach of a magnetized

needle, it is possible that the efficient cause of the disturbance may not reside in the needle; the presence of this latter, for aught we know, may be nothing more than a condition, according to which, something external to the needle is brought into action; and when we speak of the energy of the needle, it is not to be supposed that we intimate any opinion on a subject, with which we have no concern in this branch of philosophy. Nay in our investigations, we leave out much of what is discoverable, or even already known, respecting these forces. Certain bodies, under particular excitement, are known to exert such influences on other bodies; yet the peculiar properties of such bodies, or the methods by which these powers are called into action, are here overlooked, as not pertaining to the subject of abstract mechanics.

It must not, however, be supposed, that our speculations, because of their abstractedness, are incapable of any useful application. We should derive but little benefit from the force of wind or water, did we not know how to vary, at pleasure, the magnitude and direction of the impression to be received from it: how to direct a force against an obstacle capable of sustaining it, and how to economize the effort by which this is to be effected: how to modify the motion, which is the effect of the force, to convert reciprocating into rotatory motion, and *vice versâ*; and of the elements of motion, which are the mass and its velocity, how to augment either, at the expense of the other.

But the value of mechanical philosophy is to be esti-

mated, not merely by the light it affords, as to the disposal of the forces at our command: for by establishing the connexion between motions and forces, in all their modes of application, it enables us to ascend, from the motions observed in the celestial bodies, to the forces by which they are animated; and showing how to discover their intensities, and laws of action, to follow them in all their consequences, past, present, and future, of which, many would have otherwise eluded our keenest observation. It is thus that Physical Astronomy has become a science, scarcely yielding, in evidence, to abstract mathematics. The planetary system is, therein, regarded as a vast machine, which exhibits to our observation, the effects of the forces, by which these bodies are mutually influenced: and having traced up the apparent to the real motions, and these to the forces by which the whole is enlivened, we become possessed of the principle, from which we are enabled to derive, not only the motions actually observed, but all that shall occur, for ages to come, among the various bodies of this vast assemblage.

To manifest the importance of the science of Theoretic Mechanics, it is sufficient to point to the fruits it has borne in Physical Astronomy, and in Practical Mechanics, of which, the former enlarges, so wonderfully, our conceptions of the power and wisdom of the Creator; whilst the latter extends, in a manner no less wonderful, the power of man himself over the materials of this globe, which he was given to inhabit and commanded to cultivate. And it is manifest, that if ever

he shall attain to the same knowledge of the forces by which the other operations of nature are conducted, and to the same skill in applying them to his uses, it is to experimental investigation, guided and illustrated by Theoretic Mechanics, that he must be indebted for this further advancement.

To return from this digression; the problems to be solved in this branch of science, are such as the following. "To determine the conditions of equilibrium among forces simultaneously applied. To ascertain the force which results from their actions when unbalanced; *i. e.* the energy and direction, with which the body is effectively impelled; or conversely, to resolve a given force, into other forces from whose combined actions it would result. To deduce the motions which would follow from the action of certain forces applied under certain conditions; or on the contrary, to trace back certain motions to the forces by which they are produced." With respect to the forces concerned in such questions as these, there are but three particulars which claim our attention, viz.: 1. The intensity of the force; 2. Its point of application; 3. Its direction: and it is now to be shewn how these things are estimated and designated.

1. With respect to intensity, it is obvious that there are two ways of comparing forces, viz.; by means of the motions which are the effects of their unobstructed actions; and immediately, by opposing the forces to each other. Each of these methods shall be applied in its proper place; but it is evident that we should begin

with that which is the more simple, and such is the method of immediate comparison. The other method of estimating the energy of forces proceeds on certain physical principles, which therefore are to be previously established by experience. Thus; before that motion or change of motion, can be regarded as the measure of force, it must be proved that there is in matter, no innate tendency to rest or motion; and therefore, that every change in this respect, is to be ascribed to some external influence.

Again: though the motion, and consequently the velocity acquired by a body, is the effect of the force applied, and of that exclusively, yet the same thing may be asserted of the square, the cube, or of any other function of the velocity; and which of these functions is to be regarded as the measure of the force, is a question which experience alone can determine.

Further: in estimating the energy of causes by their effects, all the circumstances which influence the results, should be taken into account. If then force is to be measured by the quantity of motion produced by it, we must consider what circumstances, besides the intensity of the force, are concerned in this effect; and it will immediately occur, that time and space are so concerned. A force, however great, must continue to act during some portion of time, however short, and through some part of space, however small, in order to produce any motion whatever. Wherefore, in estimating the energy of the force by the motion produced, it would seem necessary to take into account, the time or

space through which the action is continued. Whether then, for the measure of the force, are we to take the motion generated by its action through a given space, or during a given time; and in comparing the energies of two forces, how are we to ascertain which is the circumstance that fixes the parity of condition with respect to their actions?

Such questions may be further multiplied, and they must all be solved, before we attempt to measure the force by the motion produced. In the mean time, we know that two forces are equal, which by acting on the same material point, in opposite directions, are in equilibrio: and that two equal forces, acting in the same direction, constitute a double, three a triple force, &c.: and this is quite sufficient for our purpose, when forces only are contemplated, and until we come to treat of the motions, which they are fitted to produce.

And here it is to be observed, that whatever method is taken for the measurement of force, all our estimates are comparative. This observation is applicable to quantities of whatever kind: our notion of the magnitude of any quantity is merely the notion of the relation which it bears to some other of the same kind: for which reason, forces may be represented by quantities of any other kind, the designation being understood to be only of relative magnitudes. By this commutation, much is gained, when the quantities employed are more distinctly conceived, more clearly expressed, or more easily exhibited, than those for which they are substituted: and it is evident that

numbers and lines are recommended by these advantages. Further; this comparative estimate is most readily carried on, by referring the quantities of each kind, to some fixed standard. Such a standard of comparison is unity among numbers: and if in the same way, we fix on some certain force as the unit of forces, then any force whatsoever, shall be expressed simply by the number, which is to unity, in the same ratio, as the force in question to the unit of forces. In the like manner, by fixing on the linear unit, to which all other lines are to be compared, any force shall be represented simply by the line, which bears to the linear unit, the same ratio, as the force in question bears to the unit of force.

It is by this substitution of the more abstract quantities for forces, that the latter are brought within the province of mathematics.

2. The point of application being a point of space, may be immediately exhibited to the eye, and it has its algebraical designation, by reference to three coordinate axes or planes, as in analytic geometry.

3. The direction of the force is the same with that of the line which the material point would be made to describe, if it were free to obey the impression. Accordingly, a line will serve to exhibit the direction of a force, as well as to represent its magnitude. Now when a right line passes through a given point, its direction may be determined by means of the angles which it makes with three coordinate axes: and as this is the method most frequently adopted in the following

pages, it seems proper, in this place, to offer some explanation of the principle, and of the manner in which it is applied.

Let Ax, Ay, Az, (Fig. 1.) be three rectangular axes, meeting at A, a point in the right line whose direction is to be determined. Let this line be Am, and let α, β, γ, denote the angles mAx, mAy, mAz, which the line Am makes with these axes. Now if the angle α is given, the position of the line Am, is limited to the surface of a cone whose axis is Ax, and whose vertical angle is 2α. If β is given, it is in like manner, limited to another conical surface, whose axis is Ay, and whose vertical angle 2β. Wherefore, if both α and β are given, the line Am is formed by the intersection of these two surfaces. But there are two such lines of intersection lying in a plane which contains Az, and making equal angles with the perpendicular plane yAx, and therefore making with Az, angles whose sum is 180°. The angle γ, when given, determines the line to one of these intersections.

Hence it is evident, that we are not at liberty to assign to these angles any magnitudes we please, and then require the position of the line from such data. In fact, the angle α being assumed, the angle β must be assigned between the limits $90°+\alpha$ and $90°-\alpha$; and this condition being observed, with respect to two angles, the third angle γ is restricted to a certain angle, or its supplement.

In order to perceive more clearly, the relations by which the angles α, β, γ are connected, and by which

we are, therefore, restricted in the choice of them, when from such data we proceed to ascertain the position of the line; it will be convenient to reverse the process, by regarding the line Am as given. Then if a parallelopiped is constructed, having this line for its diagonal, and for its angle, the solid angle A, made by the three axes, it is evident that the three linear sides, AB, AC, AD, are determined by the perpendiculars let fall from the point m on each of the axes, and that their magnitudes may be expressed by means of the angles as follows:

$$AB = Am.\cos.\alpha. \quad AC = Am.\cos.\beta. \quad AD = Am.\cos.\gamma.$$

Further; the square of the diagonal being equal to the sum of the squares of the sides, we have $Am^2 (= AB^2 + AC^2 + AD^2) = Am^2 (\cos^2.\alpha + \cos^2.\beta + \cos^2.\gamma)$; giving,

$$\cos^2.\alpha + \cos^2.\beta + \cos^2.\gamma = 1,$$

which expresses the condition to be observed, in assigning the magnitudes of the angles, when the position of the line is sought from these data.

With respect to the first angle α, there is no limitation, inasmuch as the square of the cosine of an angle cannot exceed unity; but the angle α being assigned, β must be such, that $\cos^2.\alpha + \cos^2.\beta$ shall not exceed unity; *i. e.* β must not be greater than $90° + \alpha$, or less than $90° - \alpha$, its supplement.

When β is taken at either of these limits, we have $\cos^2.\alpha + \cos^2.\beta = 1$: and therefore, $\cos^2.\gamma = 0$, or $\gamma = 90°$. When α and β are both given, the value of

$\cos^2.\gamma$, is $1-(\cos^2.\alpha+\cos^2.\beta)$. and γ is one or other of the supplemental angles, whose cosines are

$$+\sqrt{(1-\cos^2.\alpha-\cos^2.\beta)}, \text{ and} -\sqrt{(1-\cos^2.\alpha-\cos^2.\beta)}.$$

If the cosine of any of the three angles is negative, that angle is obtuse, and this circumstance renders the designation by cosines, peculiarly explicit, the signs prefixed, serving to identify, among the eight solid angles contained by the three coordinate planes, that which contains the line Am. Thus if $\cos.\alpha$ is negative, the line Am. falls at the side of the plane zAy opposite to that in which it should have been found, had the cosine of that angle been affirmative, *i. e.* the abscissæ taken in the direction of Ax, being deemed affirmative, the lines Am, Ax, in the case here supposed, lie at opposite sides of the plane zAy. The same is to be observed with respect to the cosine of the angle β and the plane zAx, and of the cosine of the angle γ and the plane xAy.

If the line is in a given plane, two axes will be sufficient, by taking them in the given plane. Thus if the plane were xAy, we should have $\gamma=90°$ and $\cos.\gamma=0$; and then, the equation of condition, to be fulfilled in the choice of the angles, would be $\cos^2.\alpha+\cos^2.\beta=1$.

If the line is parallel to a given right line as AB, the latter may be taken for an axis, and the condition of parallelism is expressed by the equation, $\cos^2.\alpha=1$: in which case the angle α is either cipher or 180°.

If the line does not pass through A, the origin of

the coordinates, the angles by which its direction is determined, are those which it makes with three lines drawn through any point of the same, parallel to the three axes; and by these angles together with the coordinates of the assumed point, the position of the line is completely determined.

It has been already stated, that forces may be compared together without recurring to the motions which they are fitted to produce. Indeed, in numberless cases, these effects do not follow, the forces being either wholly or in part counteracted: and in this latter case, the motion produced, is that due to the unbalanced force, *i. e.* to the force which remains, after deducting those forces or parts of forces which are in equilibrio. Thus when a body is supported by the hand, and thereby prevented from descending towards the earth's surface, a pressure is felt, which is then the entire weight of the body. If the hand were to descend under its load, the pressure would be suitably diminished; but the hand is not entirely relieved from the pressure of the body in contact, unless it descends with the celerity with which the body falls, when free to obey the force of gravity. In these instances, the pressure is the part of the weight counteracted, and the motion of the body downwards is the effect of the remainder.

From this simple instance, it will appear requisite to ascertain the forces which are so counteracted, before we can properly proceed to reason about the motions produced.

Hence the science of mechanics naturally resolves itself into two branches; that which treats of forces independently of motion; and that wherein the motions are deduced from the exciting forces, and *vice versâ*. The one branch, relating chiefly to cases of equilibrium, is denominated Statics. The other, relating to the effects of unbalanced forces, is named Dynamics.

The theory of statics naturally precedes that of dynamics because of its greater simplicity. It is altogether independent of the consideration of motion, and therefore of time and space; as also of the mass of the body moved and its inertia. In this branch of mechanics, the body acted on, is regarded, merely as an assemblage of points to which certain forces are applied; and its properties are considered, no farther than as it is more or less fitted to transmit these forces from one point to another. Further: though gravity is a force by which every particle of matter is affected, it will be found convenient to establish the more general theorems without reference to this force, leaving it to be regarded as part of the system of forces concerned in the particular cases, to which these theorems are to be applied.

The science of dynamics, having for its end, the establishment of the connexion between forces and motions, under all the circumstances in which they can be offered to our thoughts, must rest upon a foundation of certain physical principles, denominated the laws of motion: which as they seem not to be established by an inherent necessity, are to be collected from observation and experiment.

Because of the peculiar modification of mechanical action, which belongs to the constitution of fluids, it is found convenient to separate the mechanics of fluid, from those of solid substances: and adopting a like distinction in this part of the subject, we give the name of Hydrostatics, to that which treats of the equilibrium of fluids; and of Hydrodynamics to that which treats of their motions.

Gaseous substances are distinguished from other fluids, by the properties of compressibility and elasticity; and accordingly, these become the subject of another branch of mechanical philosophy, denominated Pneumatics.

The Reader is requested to make the following Corrections.

Page 207, five lines from bottom, after the word "curve" insert the words "of the intrados;" and in the next line, in place of the word "it" insert "that of the extrados."

—— 349, at the beginning of the third paragraph insert the words "If the motion is uniform."

—— 293, 296, 297, insert 7, 8, 9, the numbers of the Articles.

—— 309, 313, 317, for 3, 4, 5, read 5, 6, 7, do.

STATICS.

SECTION I.

OF FORCES APPLIED TO A MATERIAL POINT.

1. As all reasoning consists in connecting certain propositions with others previously received as true, it is requisite that we should begin with those which lie nearest to first principles: and such are the propositions which immediately follow.

"As forces in equilibrio destroy, each the effect of the rest; the body, by means of which they are opposed to each other, is in the same condition, as to rest or motion, as if they had not been applied."

Hence, we are warranted, in introducing, or suppressing any system of forces in equilibrio: and in the course of our demonstrations, it will frequently be found convenient to resort to this artifice.

"If a system of forces is in equilibrio, the equilibrium shall not be disturbed, by fixing a point in the body to which they are applied."

For if the forces engaged are in equilibrio, there is no tendency to motion thence resulting, and therefore no pressure on the fixed point, and consequently, no reaction; *i. e.*

there is no new force introduced into the system, by fixing one or more points of the body.

"Two equal forces, applied to the same material point, in opposite directions, are in equilibrio."

For no reason can be assigned, why either should prevail: *i. e.* why the material point should be moved in the direction of one of the equal forces, rather than in that of the other.

Also, "Two equal and opposite forces are in equilibrio, though they should not be immediately applied to the same point, provided, that the line connecting the points of application is of invariable magnitude." For then neither of the points can be moved in the direction of the force applied to it, without drawing the other along with it, and there is no reason, why this common motion should be in the direction of one of the forces, rather than in that of the force equal and opposite.

Hence it follows, that when the system of points is invariable, such as those of a perfectly rigid body, we are at liberty to change the point of application of a force, for any other in the line of its direction.

Thus let the force p applied at the point m, act in the direction BA (Fig. 2.) this force may be transferred to any other point as m' in the same line. For, if at this latter point, we apply two forces, acting in the same line, but in opposite directions, and each of them equal to p, this will have no effect. There are then three equal forces engaged, viz. one acting at m, in the direction of mA; a second at m' in the direction of m'B, and a third at the same point in the direction of m'A. Of these, the first and second are in equilibrio, and may therefore be suppressed; after which, there remains but the third, which is the original force p, whose point of application is transferred from m to m', a new point in the line of its direction.

Two forces not immediately opposite in direction, cannot be in equilibrio.

Let the two forces act in the directions of the different lines AB, CD, (Fig. 3.) If these forces could be supposed to be in equilibrio, that equilibrium would not be disturbed, by making the system of points invariable, or by fixing one of those points as *m* taken in the line AB, the direction of one of the forces: this being done, the force which acts in the direction of this line, will evidently be destroyed by the reaction of the fixed point; wherefore, the force in the direction of CD will then remain alone, and will turn the system round the fixed point *m*. There is then, no equilibrium when this point is fixed; and therefore, none when it is free. Hence it follows that

"If two forces are in equilibrio, they must be directly opposed, and therefore, they must also be equal."

When two or more forces not in equilibrio, are applied to the same material point, they must give to that point, a tendency to move with a certain velocity, and in a certain direction. The effect then, is the same, as that of some one force. This last is named the resultant of the forces actually applied, and they its components. Generally, the resultant of any forces is that force, which would produce alone the same motion which results from the combined action of the original forces: and the components of any force, are those from whose combined actions the same motion would result: wherefore, in all statical inquiries, the components may be replaced by their resultant, and *vice versâ*.

Accordingly, when a system of forces is applied to a point, and not in equilibrio, the equilibrium shall be established, by applying a new force, equal and opposite to their resultant.

Conversely, if a system of forces is in equilibrio, any one of them, its direction being changed into the opposite, is the resultant of the rest. For any one of them is in equilibrio with the resultant of the rest, and therefore equal and opposite to that resultant.

2. To find the resultant of two or more forces applied to the same material point, is the fundamental problem of Mechanics. The most simple case of this problem, is that in which the forces act in the same line.

If two forces p and p' are applied to the same material point, and in the same direction, their resultant is equal to their sum, *i. e.* $\text{R} = p + p'$.

This is manifest, without looking for the measures of the forces in the motions they are fitted to produce.

If two forces p and p' are applied to the same material point, but in opposite directions, their resultant is equal to their difference, and acts in the direction of the greater force, *i. e.* supposing p to be the greater force, we shall have $\text{R} = p - p'$.

This will be evident, by resolving the greater force p, into two, one part equal to p', and the other to $p - p'$. For then we have three forces, viz. p', $p - p'$, acting in the same direction; and p', acting in the opposite direction: of these, the first and last are in equilibrio; and these being suppressed, there remains only the force $p - p'$, which is therefore the value of R.

These things are equally true of forces acting in the same line, though not immediately applied to the same point; the line being supposed to belong to an invariable system; and they are extended to any number of forces acting according to such a line, by stating, that the general resultant of the forces is equal to the difference between two sums, viz. the sum of those which act in one direction, and the sum of those which act in the opposite direction, the direction of the resultant being that of the greater sum.

This proposition may be announced, in a manner still more compendious, if having prefixed positive signs to the symbols of the forces acting in one direction, we mark with negative signs, those which act in the opposite direction: for then we may state, that the resultant of all the forces acting

in the same line, is equal to their sum; using this term in the same sense as in common algebra. Thus if three forces $+p, +p', +p''$ act in the same direction, and two forces $-p''', -p''''$ in the opposite direction, we may assert, that the general resultant is equal to their sum, paying attention to their signs; *i. e.* $R = p + p' + p'' - p''' - p''''$.

To find the resultant of two forces applied to the same material point, when their directions are oblique to each other, is a more difficult problem. We shall proceed to its solution by degrees, first ascertaining the direction of the resultant, and then its quantity.

The direction of the resultant must be in the plane of the components. For, if it is supposed to be that of a line on one side of this plane; another line may be assigned, symmetrically situated, on the other side of the same plane; and whatever can be supposed to determine the resultant to one of these directions, must equally serve to give it the other. But the resultant cannot take two different directions; therefore it cannot lie at either side of the plane of the components.

The direction of the resultant of two forces applied to the same material point, must lie within the angle contained by their directions.

Let the forces act on the point A in the directions Ay, Ax. (Fig. 4.)

The former would draw the point out of the line Ax on the side of Ay; and the latter would draw it out of the line Ay on the side of Ax: wherefore the effect of both conjointly, is to give to the point a tendency to move in a direction intermediate between the lines Ay, Ax: *i. e.* the direction of the resultant must lie within the angle yAx.

If the components are equal, it can be shown, by reasoning in the same way as in the first part of this article, that the angle made by their directions is bisected by that of the resultant.

If without any change in the directions of the components, one of them is increased, the other remaining unvaried; the effect on the direction of the resultant is to diminish the angle which it makes with the component so increased.

For let the forces P and P′ act in the directions Ax, Ay, and let Az be the direction of their resultant R. (Fig. 5.) Then, if P becomes P + p, the new resultant R′ is that of R and p, and must therefore lie within the angle contained by the directions of these last forces; but the direction of p is that of P or Ax, wherefore the direction of R′ must be a line, such as Az', lying within the angle zAx.

Knowing the direction of the resultant of two equal forces, applied to the same material point, and making with each other any angle whatsoever, we may proceed to ascertain the direction of the resultant, when one of the equal forces is multiplied by any integer number.

For A, being the point at which the equal forces are applied, (Fig. 6.) and Ay, Ax, their directions, if two equal portions of these lines measured from the point A, as AE, AB, are taken to represent these forces, and the parallelogram EB is completed, its diagonal AF, which bisects the angle at A, shall be the direction of the resultant. This resultant, therefore, may be transferred from A to F, at which point, if it is resolved in directions parallel to Ay, Ax, it shall reproduce its components, viz. FQ, FG, each equal to AE or AB.

Now, if the force applied at A in the direction Ax, were made AC = 2AB, this would be adding the force AB or BC to the forces FQ, FG; and if the force FQ were transferred to B, and there compounded with BC, the resultant of these two forces would take the direction of BG parallel to AF; and the point G, where it meets the direction of the force FG, would be a point in the direction of the resultant of the forces AE and AC; A being necessarily another: wherefore the direction is that of the connecting right line, which is

the diagonal of the parallelogram, whose sides are AE and AC.

In the same manner, if at the point G, the resultant of the forces AE and 2AB, is resolved in directions parallel to Ay Ax, it shall reproduce these components, viz. GR = AE and GI = 2AB: and if the former of these is transferred to the point C, and there compounded with another force equal to AB acting in the direction Ax or Cx, the resultant shall be parallel to AF, and the point H, where it meets the line GI, shall be a second point in the direction of the resultant of the forces AE and 3AB: moreover, the line EH being equal to 3EF = 3AB, this direction shall be that of the diagonal of a parallelogram whose sides are AE and 3AB. This method of investigation will serve for mAB any multiple of the force AB.

In like manner, if the resultant of the forces AE and m.AB, is transferred to the further extremity of the parallelogram formed by these lines, and is there resolved in directions parallel to Ay, Ax, it shall reproduce these components: and if m. AB is transferred from this point to E, and there compounded with another force EP = AE, acting in the direction Ey, it shall give a resultant parallel to the diagonal of the parallelogram whose sides are AE and m.AB: and the point where it meets the side parallel to Ay, being a second point in the direction of the resultant of 2AE and m.AB, this direction is that of the diagonal of the parallelogram formed by these lines. Thus by treating the force m.AB, as the force AE was treated in the former part of this investigation, the force AE may be multiplied by any integer number n, and it may be shown, that the direction of the resultant of the forces represented by n.AE and m.AB, is that of the diagonal of the parallelogram formed by these lines.

From this it appears, that when two commensurable forces are applied to a material point, making with each other any angle whatsoever, the direction of their resultant is that of

the diagonal of a parallelogram, whose sides are taken in the directions of the forces, and in the same ratio. For if two forces, each equal to the common measure, were to act at the same point and in the same lines, the direction of their resultant would be that of the diagonal of a parallelogram, formed by equal portions of those lines; and when the forces are multiplied by any integer numbers, the direction of the resultant has been found to be that of the diagonal of a parallelogram, whose sides, taken in the same directions, are had from the former by the same multiplications.

Finally: as the unit of force may be taken of any magnitude, however small, the proposition may be extended to the case where the forces are incommensurable. For let two such forces applied to the point A, be represented in quantity and direction by the lines AB AC, and completing the parallelogram AE, (Fig. 7.) let a portion EF, be taken on the line EB, measured from the point E; and however small this portion, it is evident that a submultiple of AB, may be found less than it; and that this submultiple, taken repeatedly from the line BE, shall have one of its terminations at some point as *o*, intermediate between F and E; then drawing *o*G parallel to BA, we have a parallelogram BG, whose sides are commensurable; and therefore, the resultant of the forces, represented by AB, AG, shall have the direction A*o*, which makes with AC, an angle less than the angle FAC. But the resultant of the forces AB, AC, makes with AC, an angle less than *o*AC, and therefore *a fortiori*, less than FAC. Again: taking from the point E, on the line EC, a portion EH however small, it can be shewn in the same way, that the resultant of the forces AB, AC, makes with AB, an angle less than the angle HAB. This resultant therefore, falls within the angle FAH, however small, which can be true of no line, but the diagonal AE.

It is now proved, that when two forces of any magnitudes, are applied to the same material point in directions

making any angle whatsoever, the direction of their resultant is that of the diagonal of a parallelogram, whose sides are taken in the directions of the component forces, and proportional to them in magnitude.

But further: the resultant of two such forces, is represented by the same line, in magnitude also: For let the forces be represented in magnitude and direction, by the lines AB, AC; (Fig. 8.) and let R denote the resultant whose magnitude is to be determined. Completing the parallelogram contained by the lines AB, AC, the direction of R is that of the diagonal AE. And as the resultant, its direction being changed into the opposite, is in equilibrio with its components, it follows, that if EA is produced beyond the point A, as to G, a force equal to R, acting in the direction AG, shall be in equilibrio with AB, AC. Again: any one of the equilibrating forces, its direction being changed into the opposite, is the resultant of the remaining forces; accordingly, producing CA in the direction AM, a certain force acting in the direction AM, is the resultant of the force AB, and of the force R acting in the direction AG. Wherefore, AM coincides with the diagonal of a certain parallelogram, one of whose sides is AB, and the other is in the direction of AG. This is sufficient to determine the parallelogram; for if from B, a line is drawn parallel to AG, and from the point H, where it meets the diagonal, another line, HI, is drawn parallel to BA, the parallelogram is completed. Now the force equal to R, and acting in the direction AG, must be represented by AI: for a force in this direction, if greater or less than AI, would not compound with AB, a force in the direction of AH, the diagonal of the parallelogram. But IA = HB = AE, which completes the proof that "the resultant of two forces, acting on a material point, in directions making any angle whatsoever, is represented in quantity and direction by the diagonal of a parallelogram, whose

sides represent the component forces, in quantity and direction."*

3. By substituting for one side of the parallelogram of forces, the opposite side which is equal and parallel, it will follow, that any two forces meeting at a point, and their resultant represented by the diagonal, are proportional to the sides of one of the triangles, into which the parallelogram is divided by this diagonal; and consequently, to the sides of any triangle, whose sides are parallel to the directions of the three forces.

When the triangle is used in place of the parallelogram,

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* If it be admitted that force is measured by the velocity which it generates in a given body, by its action during a given time, this theorem may be demonstrated in the following manner.

If a body moving in the right line AC (Fig. 9.) receive an impulse in a direction *mo* perpendicular to that line, such an impulse does not alter the velocity with which it recedes from the line AB also perpendicular to AC. For if it did, an equal and opposite impulse *mo'* should double the change, since they are symmetrical with respect to AC: but they are in equilibrio, and therefore produce no joint effect.

Now let the body receive, at the same moment, two impulses, which acting separately, would carry it in the same given time over the lines OC, OD, perpendicular one to the other. (Fig. 10.) At the end of that time, the body shall be found at E, the extremity of the diagonal of the rectangle formed by the lines OC, OD. For the impulse OD perpendicular to OC, does not alter the velocity with which the body recedes from the perpendicular OD, by virtue of the impulse OC: therefore, at the end of the given time, the body shall be found at the same distance from OD, *i. e.* somewhere in the perpendicular CE, whether OD acts or not. For the same reason, by virtue of the impulse OD, at the end of the given time, the body shall be found at the same distance from OC, whether the force in OC acts or not, *i. e.* somewhere in the perpendicular DE. Wherefore, since at the same moment, it arrives at both lines CE, DE, it must arrive at E, the only point common to both. Moreover, as the body after leaving the point O, is not supposed to receive any other impulse, its motion from O to E must be uniform and rectilinear: accordingly in the given time, it must describe the right line OE, which therefore represents the compound force both in quantity and direction.

Now when the impulses AB, AD make any other angle, (Fig. 11.) the impulse AB is resolvable into the two at right angles, A*o* and A*m*, ∴ we have two impulses, A*o*, and $AD + Am = AD + Dn = An$, which, being perpendicular one to the other, compound the impulse represented by AC, the diagonal of the parallelogram *on*, and which is also the diagonal of the parallelogram BD.

it is to be observed, that the angle contained by the sides which represent the component forces, is the supplement of that contained by the forces themselves. And this observation is to be attended to, when from the component forces and the angle contained by them, the resultant is to be determined, either graphically or by computation.

The component forces and their resultant, being thus represented in quantity and direction, it follows, that all questions relative to the magnitudes and directions of these forces, are reducible to those which relate to the sides and angles of a triangle.

Thus, the sides of a triangle, being proportional to the sines of the opposite angles, and the sine of an angle and of its supplement being the same, these analogies will subsist among the forces and the sines of the angles contained by their directions. These analogies may be thus expressed,

$$\frac{\text{P}}{\sin.\,\theta'} = \frac{\text{P}'}{\sin.\,\theta} = \frac{\text{R}}{\sin.(\theta + \theta')},$$

wherein, the component forces, and their resultant, are denoted by P, P′, R, and the angles contained between each of the former and R, by θ, θ'; and, therefore, that which they make with each other by $\theta + \theta'$.

The foregoing equations express the proportionality of the forces to the sines of the angles, each force being analogous to the sine of the angle contained between the directions of the remaining two. From which it appears, that any two of the three forces, are reciprocally proportional to the sines of the angles which they make with the direction of the third force; or that the products are equal, which are obtained by multiplying each of the two forces into the sine of the angle which it makes with the third force.

As the perpendiculars let fall on the directions of two forces, from any one point in the direction of the third, are proportional to the sines of the angles they subtend, the same things may be expressed, by stating, that any two of

the three forces, are to each other, reciprocally as the perpendiculars on their directions, let fall from any one point in the direction of the third force; or that the products are equal, which are obtained by multiplying each of the two forces into the perpendiculars on their directions, let fall from any one point in the direction of the third force.

Accordingly, the angles being given, with any one of the three forces; or two of the forces and the angle contained between one of them and the third, the remaining force and angles are found by these analogies. The second of these cases is, however, subject to the same ambiguity, as in the solution of a triangle: for the angle is sought by its sine; and the sine of an angle is also that of its supplement.

When the components are given, together with the angle contained by them, the value of the resultant is known from the equation

$$R^2 = P^2 + P'^2 + 2PP'.\cos.(\theta + \theta').$$

The last term of this equation is affected with a positive sign, because the angle $(\theta + \theta')$, is the angle of the parallelogram at the point where the forces are applied, and this angle is the supplement of the angle of the triangle opposed to the side representing R.

4. The parallelogram of forces, serves equally for the resolution, as for the composition of forces; but the problem is indefinite, when nothing is given but the force to be resolved; as appears from the foregoing equations, each of which, contains four quantities, and therefore determines the value of one of them, only when the three others are given. And without resorting to these equations, it is evident, that any line, by which the force is represented, may be made the diagonal of numberless parallelograms; and therefore, that any given force may be resolved into numberless pairs of forces.

The data most commonly given in the resolution of forces, is the force to be resolved, and the directions of the compo-

nents. It is often required to resolve a force into two at right angles: *i. e.* into one in a given line and another perpendicular to same. The force in the given line shall be $P.\cos.\theta$; θ being the angle made by P with that line; and the force perpendicular to given line is $P.\sin.\theta$, as may appear by completing the rectangle.

5. By applying to the same point, a third force, equal and opposite to the resultant of two forces, the system is reduced to equilibrium. Wherefore, "if to a material point, three forces are applied, proportional to the three sides of the triangle to which they are parallel, they shall be in equilibrio."

In estimating the directions of the three equilibrating forces by those of the sides of the triangle, the sides must be taken consecutively, *i. e.* each is to be considered as directed from the last named point; as when they are taken in the order *ab*, *bc*, *ca*, (Fig. 12.) or in the contrary order, *ac*, *cb*, *ba*. Such forces are in equilibrio; for *ab*, and a force, equal and parallel to *bc*, applied at the point *a*, give the resultant *ac*, to which the third force *ca* is equal and opposite: also *ab* with a force equal to *ca*, taken in this line produced beyond *a*, compound a resultant equal and opposite to *bc*, considered as acting at the same point.

Hence all that has been demonstrated from the properties of a triangle, relative to two forces applied to a point and their resultant, is equally applicable to three forces in equilibrio about the same point.

6. A system of forces applied to a material point, and whose directions, measured from that point, lie all at the same side of a plane passing through the same point, must have a resultant: and this will be true, whatever be the number of such forces in the system.

Let the forces be *op*, *op'*, *op''*, *op'''*, &c. (Fig. 13.) all directed towards the same side of the plane *ab*, passing through the point *o*. The forces *op*, *op'*, have a resultant

which lies in the plane of these forces, within the angle contained by them, and therefore, at the same side of the plane *ab*. The same thing is true of their resultant R and another of the forces as op''; of their resultant R′ and another of the forces op'''; and so of any number of forces acting in this manner.

If three forces act on a material point, but not all in the same plane, and are represented by three lines measured from the common point and in the same directions, the resultant shall be in the direction of the diagonal of the parallelopiped having these lines for its sides, and it shall also be represented in magnitude by the same line.

For let AB, AD, AF, represent the component forces, (Fig. 14.) and let ABCDEFGH, be the parallelopiped; whose diagonal passing through the point A is AH. The resultant of the forces AB, AD is AC, the diagonal of the parallelogram whose sides are AB, AD; and the resultant of AC, AF, is AH, the diagonal of the parallelogram whose sides are AC, AF.

Conversely; a force represented in quantity and direction, by the diagonal of a parallelopiped, is resolvable into three forces, represented in quantity and direction by the sides of that parallelopiped.

7. The method of composition and resolution by means of the parallelogram of forces, may be extended to any number of forces applied to the same material point, and in any given directions, whether in the same plane or not. For as any number of forces may be replaced by their resultant, it follows, that the resultant of two forces being compounded with a third, gives the resultant of the three forces, and that this resultant, being compounded with a fourth of the given forces, gives the resultant of these four; and that the resultant of all but one, compounded with this one, gives the general resultant of the whole system of forces. These successive compositions may be effected by the construction of so many parallelograms.

Similarly, a given force may be resolved into any number, acting at the same point in its direction: first, by resolving it into two; then either or each of these into two others; and so on without limit. And it is evident, that each of these resolutions may be effected in any plane containing the direction of the force to be resolved.

8. The resultant of a number of forces applied to the same material point, may be found more expeditiously by a method founded on the triangular scheme. This is done, simply by describing a polygonal figure, whose sides are proportional to the forces, and parallel to their directions. This being done, the line which closes the polygon gives the general resultant in quantity and direction. Thus let op, op', op'', op''', op'''', represent the forces applied at the point o, (Fig. 15.) then, from p, drawing the line pq, equal and parallel to op'; the line qr equal and parallel to op''; the line rs equal and parallel to op'''; the line st equal and parallel to op''''; the general resultant of the forces shall be ot. For the force op, and the force op', to which pq is equal and parallel, shall have for their resultant the force oq: this last and op'', to which qr is equal and parallel, shall have for their resultant the force or: this and the force op''', to which rs is equal and parallel, shall have for their resultant the force os: and finally, this and the force op'''', to which st is equal and parallel, shall have for their resultant the force ot; which, accordingly, is the resultant of all the forces applied at the point o.

If an additional force is applied at the point o, represented by to, this being equal and opposite to the resultant of the rest, the whole system of forces would be in equilibrio, and the point o would not be disturbed by them: whence it follows, that any number of forces applied to a material point are in equilibrio, when the polygon is closed, by whose sides they are exhibited in quantity and direction; which indeed is evident, by considering, that as the general resultant is represented by the side which closes the polygon,

there can be no resultant when the polygon is closed by the lines representing the forces.

9. The method of graphical delineation is well fitted to assist the imagination; but when the resultant is to be made out by computation, its quantity and the angles which determine its direction should be expressed as functions of the component forces, and of the angles which determine their directions. These expressions are most conveniently obtained, by resolving each of the component forces according to three rectangular axes; then adding into one sum the forces which act in the same line, and finally compounding these three sums.

Thus let p, p', p'', p''', &c. be the forces to be compounded; A their point of application; (Fig. 16.) Ax, Ay, Az, the three rectangular axes; a, a', a'', a''', &c. the angles made by the directions of the several forces with the axis Ax; β, β', β'', β''', &c. the angles which they make with the axis Ay; and γ, γ', γ'', γ''', &c. those which they make with the axis Az. Then any force p, resolved according to these three axes, shall give the forces $p.\cos.a$, $p.\cos.\beta$, $p.\cos.\gamma$; the other forces are resolved in like manner. Wherefore, the sum of the forces acting according to the line Ax, is

$$p.\cos.a + p'.\cos.a' + p''.\cos.a'' + p'''.\cos.a''', + \&c.$$

The sum of the forces acting according to Ay, is

$$p.\cos.\beta + p'.\cos.\beta' + p''.\cos.\beta'' + p'''.\cos.\beta''', + \&c.$$

And the sum of those acting in the line Az, is

$$p.\cos.\gamma + p'.\cos.\gamma' + p''.\cos.\gamma'' + p'''.\cos.\gamma''', + \&c.$$

The whole system of forces is thus reduced to three forces acting at right angles, which may be denoted by the symbols X, Y, Z; and as the relations of these three forces to their resultant, are the same as those of the lines by which they are represented to the diagonal of the parallelopiped having these lines for its sides, the magnitude of the general resultant is given by the equation

$$R = \sqrt{(X^2 + Y^2 + Z^2)}. \quad (1)$$

The direction of the resultant is to be known by the angles which it makes with each of the three axes, Ax, Ay, Az: these angles being denoted by A, B, Γ, are given by the equations

$$\cos. \text{A} = \frac{\text{X}}{\text{R}}. \quad \cos. \text{B} = \frac{\text{Y}}{\text{R}}. \quad \cos. \Gamma = \frac{\text{Z}}{\text{R}}. \quad (2)$$

10. The equations of the line of direction of R, are those of its projections on the coordinate planes. Its projection on the plane of zx makes with the axis Ax, an angle whose tangent is $\frac{\text{Z}}{\text{X}}$. Its projection on the plane of xy, makes with the axis Ay, an angle whose tangent is $\frac{\text{X}}{\text{Y}}$. and its projection on the plane zy, makes with the axis Az, an angle whose tangent is $\frac{\text{Y}}{\text{Z}}$.

Therefore, the equations of this line are the following:

$$z = \frac{\text{Z}}{\text{X}}.x. \quad x = \frac{\text{X}}{\text{Y}}.y. \quad y = \frac{\text{Y}}{\text{Z}}.z.$$

These are to be regarded, only as two independent equations; inasmuch as any one of the three is derived from the other two.

If the origin of the coordinates is not taken at the point to which the forces are applied, let the coordinates of this latter point be denoted by $x_{,}$, $y_{,}$, $z_{,}$; and the equations of the line of direction of the resultant will be any two of the three following:

$$\left.\begin{array}{l} z - z_{,} = \dfrac{\text{Z}}{\text{X}}(x - x_{,}), \text{ or } \text{X}(z - z_{,}), = \text{Z}(x - x_{,}) \\[2ex] x - x_{,} = \dfrac{\text{X}}{\text{Y}}(y - y_{,}), \text{ or } \text{Y}(x - x_{,}), = \text{X}(y - y_{,}) \\[2ex] y - y_{,} = \dfrac{\text{Y}}{\text{Z}}(z - z_{,}), \text{ or } \text{Z}(y - y_{,}), = \text{Y}(z - z_{,}) \end{array}\right\} \quad (3)$$

11. If the forces p, p', p'', p''', &c. act, all in the same plane, this may be taken for one of the coordinate planes, as for example for the plane xy. In this case, we have

$$Z = 0. \quad \cos. \Gamma = 0.$$

whereby equation (1) becomes

$$R = \sqrt{(X^2 + Y^2)}.$$

and the direction of the resultant is determined by either of the two equations $\cos. A = \frac{X}{R}$. $\cos. B = \frac{Y}{R}$: or by the second of the equations (3): and there is the same reduction when the portions of the forces which act in the direction Az are in equilibrio; for then,

$$p\cos.\gamma + p'\cos.\gamma' + p''\cos.\gamma'' + p'''\cos.\gamma''' + \&c. = 0:$$
$$i.e.\ Z = 0.$$

If $Z = 0$, $Y = 0$, the direction of the resultant is parallel to the axis Ax: and agreeably to this, equation (1) will become $R = X$: and equations (2) will be $\cos. A = 1$. $\cos. B = 0$. $\cos. \Gamma = 0$.

In this case, the forces $p\cos.\beta$, $p'\cos.\beta'$, $p''\cos.\beta''$, $p'''\cos.\beta'''$, &c. must be either severally equal to cypher or else they must be in equilibrio: And the same is true of the forces $p\cos.\gamma$, $p'\cos.\gamma'$, $p''\cos.\gamma''$, $p'''\cos.\gamma'''$, &c.

12. For equilibrium among the forces of the entire system, we must have

$$R = 0;\ i.\ e.\ \sqrt{(X^2 + Y^2 + Z^2)} = 0.$$

And as the quantities within the parenthesis are essentially positive, this condition cannot be fulfilled, unless those quantities are, separately, equal to cypher.

The same thing appears by referring to Articles 6 and 1. In Article 6 it was shewn that three forces meeting at a solid angle, cannot be in equilibrio; and if one of the three components is cypher, the same thing has been proved of the two remaining forces in Art. 1. It appears then, by those

articles, that for $R = 0$, we must have the three forces X, Y, Z, severally equal to cypher, *i.e.*

$$X = 0, \quad Y = 0, \quad Z = 0.$$

which three equations in their more expanded form are,

$$\left.\begin{array}{l} p.\cos.\alpha + p'.\cos.\alpha' + p''.\cos.\alpha'' + p'''.\cos.\alpha''' + \&c. = 0. \\ p.\cos.\beta + p'.\cos.\beta' + p''.\cos.\beta'' + p'''.\cos.\beta''' + \&c. = 0. \\ p.\cos.\gamma + p'.\cos.\gamma' + p''.\cos.\gamma'' + p'''.\cos.\gamma''' + \&c. = 0. \end{array}\right\} \quad (4)$$

Such are the conditions of equilibrium, among a system of forces, applied to a material point, considered as being at liberty to move in any direction: and if these conditions are fulfilled, it is evident that the equation $R = 0$ is also fulfilled, *i. e.* that the system is in equilibrio.

If the point, to which the forces are applied, is confined to a surface, one of the coordinate axes, as for example, the axis Az, may be taken perpendicular to the surface at the point; and consequently, the other two axes in the tangent plane. Then

$$p.\cos.\gamma + p'.\cos.\gamma' + p''.\cos.\gamma'' + p'''.\cos.\gamma''' + \&c.$$

shall express the perpendicular pressure on the surface; with which pressure the reaction of the surface is in equilibrio. Denoting this reaction by $p_{,}$, and including this among the forces acting in the line Az, the equation

$$p_{,} + p.\cos.\gamma + p'.\cos.\gamma' + p''.\cos.\gamma'' + p'''.\cos.\gamma''' + \&c. = 0.$$

will be necessarily fulfilled. Accordingly, in the case here considered, there will be equilibrium, whenever the two first conditions are satisfied.

If the point is confined to a certain line, two of the three coordinate axes, as for example, the axes Az, Ay, may be taken perpendicular to this line, *i. e.* perpendicular to the tangent at the point. The third axis, Ax, will be then this tangent. The forces estimated according to the axes Az, Ay, and consequently the resultant of these forces shall be per-

pendicular to the line to which the material point is restricted, and therefore shall be destroyed by its reaction, *i. e.* it shall be in equilibrio with this reaction. Putting $p_{,,}$ for the reaction in the line Ay and $p_{,}$, as before, for that in the line Az, the equations,

$$p_{,,}.+p.\cos.\beta+p'.\cos.\beta'+p''.\cos.\beta''+p'''.\cos.\beta'''+\&c.=0.$$
$$p_{,}.+p.\cos.\gamma+p'.\cos.\gamma'+p''.\cos.\gamma''+p'''.\cos.\gamma'''+\&c.=0.$$

shall be necessarily fulfilled; and therefore in this case there will be equilibrium whenever the remaining condition is satisfied, expressed by the first of the three equations. (4)

SECTION II.

OF THE COMPOSITION OF PARALLEL FORCES.

1. WHEN two forces act on an invariable system, in parallel directions, and towards the same side of the line connecting their points of application, the case may be regarded as included in that of forces meeting at a point; by supposing this point to have receded to an infinite distance.

Thus, let m, m', (Fig. 17.) be the points of application of the forces P, P', acting in the same plane, and towards the same side of the line mm': and first, let their directions be mA, m'A', inclined to each other in the angle H. Then, if without making any change in the direction of the force P, we suppose the direction of the force P' to turn round the point m' in the same plane, so as to assume, successively, the positions m'A', m'A'', m'A''', &c. and finally, to coincide with m'A''''', parallel to mA, which is the direction of the force P, it is evident, that during this revolution of the line m'A', it shall meet the line Am, successively, at the points H, H', H'', H''', &c. and that when the direction of the force P' coincides with the line m'A''''' parallel to mA, the distance at which they meet becomes infinite.

During the progress of the system of forces P, P', towards this state, they had a resultant continually increasing in magnitude: therefore, the same forces, when parallel, have a resultant, whose magnitude and direction are those of the limit to which the resultant of the forces had been approaching, whilst the angle made by their directions was gradually diminished. Now, the angle of a parallelogram being diminished to cypher, the diagonal passing through that angle

increases towards a limit, which is the sum of the sides: wherefore, the sum of the component forces is the limit sought. Accordingly, when two parallel forces act towards the same side of a line transverse to their directions, the resultant is equal to the sum of the components.

Again: the direction of the resultant was ever intermediate between the directions of the components, whilst they met at an angle; wherefore, the direction of the resultant of two such parallel forces is intermediate between their directions, and, therefore, also parallel to the same.

Finally: whilst the forces met at an angle, the perpendiculars on the directions of the components, let fall from any point in that of the resultant, were reciprocally proportional to those forces: and when the forces become parallel, those perpendiculars lie in directum. Therefore, the perpendicular between the directions of the parallel forces is divided by the resultant into segments, which are reciprocally proportional to those forces; and the same thing is consequently true of the segments, into which a line connecting any two points in the directions of the parallel forces, is divided by the direction of their resultant.

2. But though these truths may be thus inferred, from those already established relative to forces meeting at a point, yet the great importance of the doctrine of parallel forces in the theory of mechanics, may seem to warrant, if not to require, a demonstration more rigorously exact. This may be obtained in the following manner:—Let the parallel forces P, P' acting at m, m' (Fig. 18.) and towards the same side of the right line connecting those points be represented by mA, $m'A'$. To this system of forces let there be added two equal and opposite forces $mg, m'g'$ acting at m, m' and in the connecting line produced beyond those points. By this addition no change will be made in the result of the original system of forces. Now compounding the forces mg, mA, and also the forces $m'g'$, $m'A'$, the resultant of the former pair will act in some

line within the angle gmA as mo, and that of the latter pair in some line within the angle $g'm'A'$ as $m'o'$, and these lines being no longer parallel shall meet at some point H. Transferring these partial resultants to this point, and resolving them in directions parallel to mg, mA, and $m'g'$, $m'A'$, it is evident that they shall reproduce the original components; viz. Hd, Hs equal respectively to mg, mA, and Hd', Hs' to $m'g'$, $m'A'$. Of these Hd, Hd', being equal and opposite, may be suppressed; and the resultant of the remaining two is their sum $Hs + Hs'$, or $mA + m'A'$; *i. e.*

$$R = P + P'. \quad (1)$$

It is now proved, that "the resultant of two parallel forces, acting towards the same side of the line connecting their points of application, is parallel to their directions, and equal to their sum."

To find the point c, where the line mm' is intersected by the direction of the resultant, we have, from similar triangles,

$$mc : cH :: ns : sH,$$
$$m'c : cH :: n's' : s'H.$$

giving

$$mc = \frac{cH.ns}{sH}. \qquad m'c = \frac{cH.n's'}{s'H}.$$

But $cH.ns = cH.n's'$, and therefore

$$mc : m'c :: s'H : sH :: P' : P.$$

Shewing, that "the line which connects the points of application, is divided by the resultant into segments, which are reciprocally proportional to the forces."

By compounding the last proportion, we have,

$$mc + m'c : m'c :: P + P' : P. \quad i.\,e.\ mm' : m'c :: R : P.$$

and in the same way,

$$mm' : mc :: R : P'.$$

which analogies are succinctly expressed by the following equations:

$$\frac{\text{R}}{mm'} = \frac{\text{P}}{cm'} = \frac{\text{P}'}{cm}. \qquad (2)$$

Stating, that "any two of the forces are, one to the other, reciprocally as the distances of their lines of direction from that of the third force."

3. The resultant R, its direction being changed into the opposite, is in equilibrio with P, P′. Thus cc' representing the resultant of the parallel forces mA, m'A′, (Fig. 19. No. 1.) becomes the equilibrating force, when its direction is changed into the opposite, as in No. 2; wherefore, "the conditions of equilibrium, among three parallel forces, are, 1. That the greatest should be equal to the sum of the other two. 2. That it should act in the contrary direction. 3. That it should divide the line connecting the points of application of the lesser forces into segments, which are reciprocally as those forces."

4. Having established the conditions of equilibrium among three parallel forces, the transition is easy, to the resultant of two forces, acting in parallel but contrary directions; for the equilibrating force, its direction being changed into the opposite, is their resultant. Thus m'A′ which equilibrates the forces mA, cc'', No. 2., becomes their resultant when its direction is changed into the opposite, as in No. 3. Whence it follows, that "if two parallel forces act in contrary directions, with respect to the line which joins their points of application, 1. The resultant is equal to their difference. 2. Its direction is parallel to those of the components, and corresponds to that of the greater. 3. Its point of application lies beyond that of the greater force, at a distance which, measured from that of the lesser force, is to the distance between those forces, as the greater to the difference: or which, measured from the point of application of the greater of the component forces, is to the distance between them, as the lesser of those forces to their difference."

In fact, the three cases of parallel forces here considered, differ only with respect to their names and directions; but not as to their relative magnitudes or points of application. If P and P′ act towards the same side of the line which connects their points of application, R, equal to their sum, is their resultant; and its direction corresponds to that of the forces P and P′. If the direction of R is changed into the opposite, its magnitude and point of application remaining unchanged, the three forces are in equilibrio; or any one of them is in equilibrio with the remaining two; and finally, if the direction of either of the first pair, as P′, is also changed into the opposite, its magnitude and point of application remaining unchanged, it becomes the resultant of P and R, which are then parallel and contrary forces. Wherefore, all questions, respecting the relative magnitudes and points of application of three parallel forces, whether one of them is in equilibrio with the other two, or their resultant, are to be managed by means of the same equations or analogies.

Equation (1) gives any one of the three forces when the other two are given: for $R = P + P'$, and therefore,

$$P' = R - P, \quad P = R - P'.$$

The three equations (2) contain, each of them, four quantities, viz. two forces, and two distances; and any three of those quantities being given, the fourth is immediately obtained by these equations.

Thus, if two parallel forces are given, acting towards the same side of the line which joins their points of application, these points being also given; and if it is required to find their resultant and its point of application, it is evident that the are data P, P', mm'; and that the quantities sought are R and cm, or cm'. But by equation (1), $R = P + P'$, and by equations (2), $cm = mm'. \dfrac{P'}{P + P'}$.

If it is required to resolve a given force into two parallel forces acting at given distances, the quantities given are R, cm, cm', and those sought are P, P'. But

$$P = R.\frac{cm'}{mm'}. \quad P' = R.\frac{cm}{mm'}.$$

If two parallel and contrary forces are given, as also the points of application, and it is required to find their resultant and the point to which it is applied, the data are P, R, and cm: and the quantities sought are P′, cm' or mm'. But $P' = R - P$, and

$$cm' = cm.\frac{P}{R - P}. \quad mm' = cm.\frac{R}{R - P}.$$

5. Considering the equation $cm' = cm.\frac{P}{R - P}$. it appears that if whilst R remains unchanged, P increases from nothing to R, cm' shall constantly increase: and that when $P = R$, the value of P′, which is $R - P$, is then cypher, and $cm' = cm.\frac{P}{0}$ is then infinite. This shows, that when two parallel forces, acting in contrary directions, are equal, there is no single resultant; or, which is the same thing, that there is no single force by which the equilibrium can be established. In fact, could we suppose a single resultant, there would be no reason why it should be nearer to one than to the other of the components; it must, therefore, either be at an infinite distance from both, or at equal distances between them; and if this latter case be supposed, there is no reason why it should corrrespond in direction with one, rather than with the other of the components, for all here is symmetrical.

If P still further increases, the denominator of $cm.\frac{P}{R - P}$, which is the value of cm', changes its sign; which denotes that this latter quantity is then to be measured on the transverse line, in the direction of cm: and because the absolute value of the denominator is less than P, the absolute value of cm' is greater than cm; diminishing from infinity to this limit, as P increases from R to infinite.

From the equation $P' = R - P$, it also appears, that in this case P′ is negative; or that its direction, which on the

supposition of $R>P$, corresponded to that of R; now, on the supposition of $P>R$, corresponds to that of P, all of which is but the analytical expression of the truth already stated, viz. that the resultant of two parallel and contrary forces is equal to their difference; that its point of application lies beyond that of the greater of the component forces, and that it corresponds with this force in direction, with respect to the line which connects the points of application.

6. The principles already established will direct us how to proceed in the composition of any number of parallel forces.

Thus let P, P', P'', &c. be any parallel forces applied at the points of an invariable system, whether in the same plane or otherwise; and let it be required to find the general resultant, as also its point of application.

We first compound those which act at the same side of a plane transverse to their directions: let P and P' be two such forces, we have for their resultant $R=P+P'$; again, compounding R with a third of these forces P'', we have the resultant of R and P'', *i. e.* $R'=R+P''=P+P'+P''$. Thus, the resultant of all the parallel forces, which act towards the same side of the plane transverse to their directions, is the sum of all such forces. In the same way, the resultant of all the forces, which act towards the other side of the same plane, is equal to the sum of these latter forces: the general resultant of all is the difference of these two partial resultants, and corresponds in direction with the greater: *i. e.* it is equal to the sum of those which act towards the same side of the transverse plane, minus the sum of those which act towards the opposite side of the same plane; and it corresponds in direction with that of the greater sum: so that regarding as positive those which are directed towards one side of the plane; and as negative, those directed towards the opposite side of the same plane, and understanding by the sign $+$ in the formula wherein they are combined, that the forces are to be connected with their

proper signs, as in common algebra, it may be stated, that the general resultant is equal to the sum of the parallel forces, or that R $=$ P $+$ P$'$ $+$ P$''$ $+$ &c.

The point of application is found for the resultant by the repeated use of the rule for finding that of the resultant of two component forces.

First, the point *c* (Fig. 20.) is had by dividing the line mm' in the inverse ratio of the forces P and P$'$. Then conceiving the force R ($=$ P $+$ P$'$) to be applied at this point, we connect it with the point m'', and we find c', the point of application of R$'$ by dividing the line cm'' in the inverse ratio of the forces R and P$''$, *i. e.* of P $+$ P$'$ and P$''$. Having found, in this manner, the point of application of the resultant of all the forces which act towards one side of the tranverse plane, we do the same for the forces which act towards the other side of the same plane, and the point of application being thus found for the resultant of each of the groups of parallel forces, we find that of the general resultant by the rules already delivered for two parallel forces acting in contrary directions.

If these two partial resultants are equal in magnitude, as well as contrary in their directions, the system of forces admits of no further reduction: *i. e.* it admits of no single general resultant, and therefore cannot be equilibrated by any single force.

From the method of finding the point *c*, it is evident that its position in the line mm' is altogether independent of the angle Amm': therefore, whilst the directions of the forces P and P$'$ turn round their points of application, the intensities of these forces and the parallelism still remaining, R the magnitude of the resultant, and *c* its point of application, shall continue unchanged. The same is to be observed of R$'$, the resultant of R and P$''$, or of P, P$'$, P$''$, if these three forces, their intensities and parallelism remaining, turn round their respective points of application: and the same thing is true of all the partial resultants, and, therefore, of the general

resultant when there is one: so that if the directions of all the forces of the system are simultaneously changed, the parallelism remaining, as also the intensity of the force applied at each point, the general resultant will continue of the same magnitude, and will have the same point of application. This point is denominated the centre of parallel forces, and may be defined to be that point of the body through which the general resultant always passes, whatever be the directions of the forces; the parallelism, as well as the intensities of the forces applied at the several points, still remaining.

The general resultant of any number of parallel forces being equal to the sum of those which act in one sense, minus the sum of those which act in the contrary sense, this sum or difference may be supposed to be applied at the centre of parallel forces; and, therefore, if a force equal to this is applied at the same point, and in the opposite direction, the whole system shall be maintained in equilibrio, whatever be the position of the body, or of the lines connecting its several points, with respect to the directions of the parallel forces.

7. As the centre of parallel forces frequently offers itself to our consideration in mechanical questions, it is of importance to show how its coordinates are determined, as functions of those of the points of application, and of the intensities of the forces there applied. This is done by computing its distance from three coordinate planes.

Wherefore, beginning with one of those planes, which may be that of xy. Let mo, $m'o'$, be the perpendiculars on this plane, from m, m', the points of application of the forces P, P': (Fig. 21.) and let cr be the perpendicular, from the point at which their resultant is applied. It is evident that o, r, o', the projections of these points shall lie in the right line oo'. From m, let mg be drawn parallel to this line oo',

meeting cr at e. Then, from similar triangles, there is obtained the following proportion:

$$ce : m'g \; (:: mc : mm') :: \text{P}' : \text{R},$$

wherefore,

$$\text{R} \times ce = \text{P}' \times m'g.$$

But also,

$$\text{R} \times er = \text{P} \times mo + \text{P}' \times go'.$$

And adding these equations,

$$\text{R} \times cr = \text{P} \times mo + \text{P}' \times m'o'. \qquad (a)$$

In the same way, from m'', the point of application of a third force P'', letting fall the perpendicular $m''o''$, as also from c', the point of application of R' the resultant of R and P'', letting fall the perpendicular $c'r'$, we have

$$\text{R}' \times c'r' = \text{R} \times cr + \text{P}'' \times m''o''. \qquad (b)$$

Proceeding in the same way with m''', the point of application of a fourth force P'''; and with c'', the point of application of R'', the resultant of R' and P''', we have

$$\text{R}'' \times c''r'' = \text{R}' \times c'r' + \text{P}''' \times m'''o'''. \qquad (c)$$

Then adding together the equations (a), (b), (c), we have

$$\text{R}'' \times c''r'' = \text{P} \times mo + \text{P}' \times m'o' + \text{P}'' \times m''o'' + \text{P}''' \times m'''o''' :$$

and so for any greater number of forces.

In the figure, all the forces, P, P', P'', &c. are represented as acting towards the plane xy, and all the points of application, m, m', m'', &c. are placed on the same side of that plane; but it is easily perceived, that if any of the forces act in the contrary direction, or if any of the points of application are situated on the opposite side of the plane, such forces and the distances of such points are to be marked with negative signs; and that the theorem is still true, viz. that "The sum of the products, formed by multiplying each of the component parallel forces into the distance of its point of application from any assumed plane, is equal to the single product

of their resultant into the distance of its point of application from the same plane." So that if Z denotes the distance of the general resultant R, and z, z', z'', &c. the distances of the several points m, m', m'', &c. we have

$$RZ = Pz + P'z' + P''z'' + P'''z''' + \&c.$$

or, putting for R its value, and dividing,

$$Z = \frac{Pz + P'z' + P''z'' + P'''z''' + \&c.}{P + P' + P'' + P''' + \&c.} \qquad (3)$$

By this theorem the centre of parallel forces is limited to a plane, parallel to that of xy, at the distance Z so determined. To fix it absolutely, its distance from each of the other two coordinate planes must also be determined; which is done for them as for the plane of xy. Let X be its distance from the plane zy, and Y its distance from the plane zx, the distances of the points of application of the components from these planes being in like manner denoted by x, x', x'', x''', &c. y, y', y'', y''', &c., we have

$$X = \frac{Px + P'x' + P''x'' + P'''x''' + \&c.}{P + P' + P'' + P''' + \&c.} \qquad (4)$$

$$Y = \frac{Py + P'y' + P''y'' + P'''y''' + \&c.}{P + P' + P'' + P''' + \&c.} \qquad (5)$$

These are the values of the coordinates of the centre of parallel forces, and by them its position is completely determined. And it is manifest, if the three lines, Z, X, Y, so obtained, are measured on the three axes, which are the intersections of the assumed planes, that the centre of parallel forces is at the remote extremity of the diagonal of the parallelopiped, having the same three lines for the sides of its solid angle.

If the points of application of the forces P, P′, P″, &c. lie in the same plane, this plane may be taken for one of the coordinate planes, as for example, that of xy; and this, whether the directions of the forces lie in the same plane or not.

By this disposition we shall have z, z', z'', &c. each equal to cypher; and, therefore, also $\text{z} = 0$: so that the position of the centre of parallel forces is then determined by the equations (4) and (5), which give the values of X and Y.

If the points of application are all arranged in the same right line, this line may be regarded as the intersection of two of the coordinate planes; as for example, of the planes xy, xz. We shall then have z, z', z'', &c. each equal to cypher; and, therefore, $\text{z} = 0$. Likewise, y, y', y'', &c. each equal to cypher, and therefore, also, $\text{Y} = 0$. Wherefore, the centre of parallel forces is in the same line with their points of application, viz. the axis of x: and its position in this line is determined by the single equation (4), which expresses the value of X, its distance from the plane zy, perpendicular to the right line in which the points are arranged.

In this case, therefore, it will be sufficient for the determination of the centre of parallel forces to ascertain its distance from any point, assumed in the line itself, as the origin: and this is done by means of any one of the three equations, (3), (4), (5).

For example, let it be supposed that there are five points, a, b, c, d, e, arranged along the line os, (Fig. 22.) to which points parallel forces are applied, denoted by the numbers, 5, 4, 8, 10, 2. These points may be referred to any point in the line itself, or in this line produced, as the origin. Let them be referred to the point m, and let the distances from this point be expressed by the numbers $-3, -1, +4, +6, +7$: the distances on one side of this point being deemed affirmative, as mc, md, me; and those on the other side negative, as mb, ma. The same distinction is to be made among the forces themselves, according as they tend to one side or the other, of the line os. In the above scheme all the forces are supposed to act in the same sense, and therefore to be affected with the same sign. Then, for the distance of the centre of parallel forces from the point m, the forces are

to be multiplied, severally, into the distances of their points of application from the point *m*, and the sum of these products is to be divided by the sum of the forces. In this instance we shall have

$$x = \frac{32 + 60 + 14 - 4 - 15}{29} = \frac{87}{29} = +3.$$

Wherefore, taking three units of the line from the point *m*, and in the direction of the affirmative values, we have the point *g*, which is the centre of the parallel forces.

SECTION III.

OF EQUAL AND PARALLEL FORCES ACTING ON AN INVARIABLE SYSTEM, TOWARDS OPPOSITE SIDES OF A LINE TRANSVERSE TO THEIR DIRECTIONS.

1. WHEN two parallel forces act towards opposite sides of a line transverse to their directions, those directions, though not immediately opposite, may be said to be contrary.

In last section, it was shown that two such forces, when equal, are incapable of being equilibrated by a single force. It is now to be shown how they are equilibrated, and how transformed. To avoid circumlocution, a pair of equal parallel and contrary forces shall be simply named a pair; and in all transformations of a pair of such forces, it is to be understood that the intensity of the forces, and the perpendicular distance between the lines of direction, remain unchanged, unless when the contrary is expressly stated.

A pair is in equilibrio with another pair, equal to the former, and applied to the same points in opposite directions.

This is evident, inasmuch as the forces applied at each point are in equilibrio, and the forces in equilibrio being suppressed, there remains no force to disturb the system.

The equilibrium continues when the second pair is transferred to any part of the same plane, in parallel directions.

Let one pair consist of the forces $+$ P, $-$P, acting in the directions Aa, Bb. (Fig. 23.) These are in equilibrio with the pair $-$P, $+$ P, acting at the same points in the opposite directions Aa', Bb'. Let the line Ax be perpendicular to those directions, and taking A$'$B$'$ $=$ AB, let the latter pair be

transferred to the points A′ and B′, in the directions A′*a*′, B′*b*′, parallel to the former. The lines AB′ BA′ shall be bisected, each at the same point *o*. The forces + P, + P, acting at A and B′, shall compound a force + 2P, acting at *o*; and the forces — P, — P, acting at B and A′, shall compound a force — 2P, acting at the same point *o*; and as these resultants are equal and opposite, they shall be in equilibrio.

The equilibrium continues when the second pair is transferred, in parallel directions, into any parallel plane.

Let the parallel planes be MN, M′N′, (Fig. 24.) and let the directions of the pair, thus transferred, be A′*a*′, B′*b*′. These lines being parallel to A*a*, B*b*, the perpendicular distances AB, A′B′, shall be parallel, and therefore in the same plane; and being also equal, the lines AB′, BA′ shall bisect each other, as at *o*. The two forces + P, + P, shall compound the force + 2P, acting at *o*; and the two forces — P, — P, shall compound the force — 2P, acting at the same point: and as these equal resultants are also directly opposed, they are in equilibrio.

The equilibrium continues, though the second pair is turned round in its own plane, in an angle of any magnitude.

Let the two pairs, first, act at the extremities of the same line AB, perpendicular to their directions; (Fig. 25.) and this line being bisected at *o*, let the second pair turn round this point in the same plane; and let A′*a*′, B′*b*′ be their new directions, meeting B*b*, A*a* at the points *n* and *m*. Drawing the line *mo*, the two triangles, A*mo*, B′*mo*, are right angled; and having A*o* = B′*o*, and *mo* common, the angles at *m*, as also the angles at *o* are equal: and the same is proved in the same way of the triangles A′*no*, B*no*: wherefore, the angles AOB′, A′OB, which are vertically opposed, being bisected hy the lines *mo*, *no*, these lines lie in directum. The forces + P, + P, acting at *m*, compound a force in the direction *mo*, and the forces — P, — P, acting at *n*, compound a

force in the direction *no*: and these resultants, being equal and opposite, are in equilibrio.

It is now proved, that "a pair is in equilibrio, not only with a pair of equal forces directly opposed, but with the same pair, when transferred, in parallel directions, to any other part of the same plane, or to a parallel plane, or when turned round in any of those planes, in any angle whatsoever."

If at the point which bisects the perpendicular distance between the forces of a pair, a line is raised perpendicular to their plane, this perpendicular may be named the axis of the pair: and the same things are only differently stated, when it is said, that "a pair is equilibrated, not only by a pair of equal forces directly opposed, but by the same pair, when transferred, in parallel directions, to any other point of the same axis, or to any parallel axis; or when turned round its axis in any angle whatsoever."

Moreover: after any of these changes the pair is in equilibrio with a pair of equal and opposite forces; and as all systems of forces are equivalent which are equilibrated by the same system, it follows, that "a pair may be transferred, in parallel directions, to any part of the same plane, or to any parallel plane, or be turned round in its plane, in any angle whatsoever. Or, which is the same thing, it may be transferred to any point of the same, or of a parallel axis, or be turned round any of those axes in any angle whatsoever."

2. The tendency of a pair is to give to the points of application, movements in the directions of the individual forces; and, therefore, to the system to which those points belong, considered as invariable, a movement about some axis perpendicular to the plane of the forces. The energy of the pair to produce this effect, can depend only on the magnitudes of the forces, and the perpendicular distance between their directions. The product of one of the equal

forces, into the perpendicular distance between their directions, may be named the moment of the pair: and if two pairs act in the same or in parallel planes; or, which is the same thing, on the same or on parallel axes, their moments may be said to be of the same or of contrary directions, according as they tend to turn the system in the same, or in contrary directions.

A pair is in equilibrio with another pair when the moments of the two pairs are equal and contrary.

Let one pair consist of the forces $+\text{P}$, $-\text{P}$, acting in the directions Aa, Bb; (Fig. 26.) and let their perpendicular distance, AB, be bisected at o. Let the other pair be $-\text{P}'$, $+\text{P}'$, which may be supposed, first, to act in the directions A$'a'$, B$'b'$, parallel to the former, and equidistant from the same point o. Then as $\text{P} \times \text{A}o = \text{P}' \times \text{B}'o$, the resultant of the two forces $+\text{P}$, $+\text{P}'$, shall be $+(\text{P}+\text{P}')$, acting at the point o: and as $-\text{P} \times \text{B}o = -\text{P}' \times \text{A}'o$, their resultant shall be $-(\text{P}+\text{P}')$, acting at the same point o: and these equal resultants, being directly opposed, shall be in equilibrio. Now the pair $(+\text{P}', -\text{P}')$ may be transferred to any point of the same, or of a parallel axis, or be turned round any one of those axes, in any angle, and the equilibrium shall continue.

Hence it follows, that all pairs are equivalent whose moments are equal and of the same direction.

Two pairs, whose moments are of the same direction, may be compounded into one pair, whose moment is equal to the sum of their moments, and of the same direction.

For, let one pair of forces, P, act at the distance d; and let another pair, P$'$, act at the distance d': this latter pair may be transformed into another pair of forces, $\text{P}'.\frac{d'}{d}$, acting at the distance d. The pairs, now acting at the same distance, may be made coincident; and they are then reduced to one pair of forces, $\left(\text{P}+\text{P}'.\frac{d'}{d}\right)$, whose moment is

$$\left(P + P'.\frac{d'}{d}\right) d, \text{ or } Pd + P'd'.$$

In the same way it is shown, that two pairs, whose moments are of contrary directions, may be compounded into a single pair, whose moment is equal to the difference of their moments, and of the same direction with that of the greater. And, in general, that "whatever be the number of pairs, placed on the same or on parallel axes, they may be compounded into a single pair, whose moment is equal to the difference between the sum of the moments of one direction, and the sum of the moments of the contrary direction; the direction of the moment of the resulting pair being the same as that of the greater sum."

Hence it follows, that "a pair may be resolved into any number of such pairs, whose moments, severally, are of the same direction with that of the pair to be resolved, or of the contrary direction; provided, that the sum of the former moments exceeds that of the latter, by a difference equal to the moment of the pair to be resolved."

The planes of two pairs being inclined to each other in any angle, if the moments are represented by the sides of a parallelogram, whose angle is that which measures the inclination of the planes of the pairs, those pairs may be compounded into a single pair, whose moment is represented by the diagonal of the parallelogram; and whose plane divides the angle, made by the planes of the components, in the same manner as the diagonal divides the angle contained by the sides of the parallelogram.

Let the two pairs consist of forces of the same magnitude, P; and let the forces of corresponding directions, as for example, the forces $+$ P act in the line BA, the intersection of the planes of the pairs. (Fig. 27.) Let those planes be intersected by a third plane in the lines AC, AD; and let the forces $-$P act at the points of those lines M and N. The two forces $-$P are equivalent to one force $-$2P, applied to

the point *o*, at which the line MN is bisected. The two pairs are then equivalent to one pair of forces 2P, parallel to the forces P, and applied on the line A*o*, which is the half of AR, the diagonal of the parallelogram, whose sides are AM, AN; and this is equivalent to a pair of forces P, parallel to the former, and applied on the whole diagonal AR.

The two pairs of forces, P, applied on the lines AM, AN, are then equivalent to one pair of forces P, parallel to the same, and applied on the diagonal of the parallelogram constructed with those sides.

If the forces of the two pairs are not of the same magnitude, they may be brought to this condition, by transforming one of the pairs into another of the same moment, whose forces shall agree in magnitude with those of the other pair: and if the forces of corresponding directions do not act in the line BA, the pairs may be turned round, each in its own plane, until the directions of the forces become parallel to this line; and then they may be transferred, each in its own plane, so as to satisfy this supposition.

Were the intersecting plane perpendicular to the line AB, the directions of the forces would be perpendicular to the lines AM, AN, AR; and the forces being all of the same magnitude, the moments of the pairs would be proportional to those lines. And as the angles made by the sides and diagonal of the parallelogram measure the inclinations of the planes of the parallel forces, it follows, that the angle made by the planes of the component pairs is divided by that of the resultant pair, in the same manner as the angle of the parallelogram by its diagonal.

Since the resultant pair is equivalent to its components, it follows, conversely, that the pair of forces P, applied on the line AR, is resolvable into two pairs of forces, parallel to the same, and of the same magnitude, applied on the lines AM, AN, which are the sides of the parallelogram whose diagonal is AR; and that the moment of the pair is to those of

the pairs into which it is resolved, as the diagonal to the sides of a parallelogram; the diagonal making with its sides, the angles which measure the inclinations of the plane of the pair, to those of the pairs into which it is resolved.

If a direct demonstration is required, such is readily supplied; for the pair of forces P applied on AR, is equivalent to the pair of forces 2P, parallel to the former, and applied on AO, the half of AR: and −2P acting at *o*, is equivalent to −P, −P, parallel to the same and applied, one of them at M, the other at N. The given pair is then equivalent to two pairs of forces, equal and parallel to the same, and applied on the sides of the parallelogram, of which AR is the diagonal.

The angles formed by the planes being the same as those made by their axes, the same relations of the resultant pair to its components, as to the magnitude of their moments and position of their planes, may be expressed by stating, that "if on the axes of the component pairs, two portions are taken, which, measured from the angle, are proportional to their moments; and if a parallelogram is constructed, having these lines for its sides, the diagonal shall be the axis of the resulting pair; and shall also represent the magnitude of its moment."

In constructing the parallelogram, it is to be observed, that when the corresponding forces are brought to act in the same line, the sides of the parallelogram should be measured from a point in this line, either both towards the directions of the remaining forces, or both from those directions: and that the sides thus measured, if made to describe an angle of 90°, by revolving in their plane, become the sides of the corresponding parallelogram, to be constructed with the axes.

By proceeding according to these methods, any number of pairs may be compounded into a single pair; first compounding two pairs into one, then the resulting pair with

another pair, and each resulting pair with a new pair, until all are compounded.

As two forces, whose lines of direction meet at an angle, must have a resultant, and, therefore, cannot be in equilibrio; so two pairs, whose planes meet at an angle, must have a resulting pair, and therefore cannot be in equilibrio.

When the planes of the component pairs meet at a right angle, the parallelogram becomes a rectangle; and the moments of the component pairs being denoted by X and Y, and that of the resulting pair by R, we have

$$R = \sqrt{(X^2 + Y^2)}.$$

Also, denoting by a, the angle formed by the plane of the pair whose moment is X, with the plane of the resulting pair, we have

$$\cos.a = \frac{X}{R}. \qquad \sin.a = \frac{Y}{R}.$$

If the axes of three pairs are parallel to three lines meeting at a solid angle, and if on these lines three portions are measured from the point, to represent the moments of the several pairs, the diagonal of the parallelopiped constructed with these sides, shall represent the moment of the resulting pair, and shall be parallel to its axis. For let the axes meet at the point A, (Fig. 14.) and the moments of the three pairs being represented by the lines AB, AD, AF, taken on the axes, let the parallelopiped DG be constructed; then, the two pairs, whose axes are AB, AD, and whose moments are represented by these lines, shall compound a pair, whose axis is AC, and whose moment is represented by the same line. This resultant of the two pairs, compounded with the third pair, whose axis is AF, and whose moment is represented by that line, shall give a pair, whose axis is AH, the diagonal of the parallelogram constructed with the sides AC, AF, and whose moment is represented by the same line. But AH is the diagonal of the parallelopiped.

These axes may be now transferred, each by a parallel movement, to any distance, and the resulting pair shall not be affected, as to the magnitude or direction of its moment.

Hence it follows, that three pairs, whose planes constitute a solid angle, must have a resultant; and, therefore, cannot be in equilibrio.

If the three planes intersect at right angles, the parallelopiped is rectangular; and, in this case, the moment of the resulting pair is equal to the square root of the sum of the squares of the moments of the component pairs, *i. e.* putting R, X, Y, Z, for these moments, we have

$$R = \sqrt{(X^2 + Y^2 + Z^2)}.$$

If α, β, γ, denote the angles made by the axis of the resulting pair, with those of its components; or by the diagonal with the sides of the rectangular parallelopiped, we have

$$\cos.\alpha = \frac{X}{R}. \qquad \cos.\beta = \frac{Y}{R}. \qquad \cos.\gamma = \frac{Z}{R}.$$

From the four last equations, any one of the seven quantities, R, X, Y, Z, α, β, γ, is known, if three of them are given. But among the given quantities, there must be a moment: for if the three angles, only, are given, we can determine, only, the relative magnitudes of the moments.

3. A pair may be compounded with a single force parallel to its plane, into a single force, and this single resultant shall be equal and parallel to the single component; the plane in which they are contained being parallel to the plane of the pair; and the distance between them depending on the moment of the pair.

For let $+P$, $-P$ be the forces of the pair; d their distance, and P' the single force. The pair may be transformed into another pair of forces $+P'$, $-P'$, of equal moment; and the distance between their directions will then be $d' = \frac{P.d}{P'}$. This pair may be now transferred, so that the force $-P'$

shall be directly opposed to the single force; these two being in equilibrio may be suppressed; and there will then remain the force $+P'$, which is the force of the pair so transformed and transferred, that corresponds in direction with the single force; or it may be regarded as the single force itself, transferred in a plane parallel to that of the pair, to a distance $d' = \frac{P.d}{P'}$.

Conversely; a single force may be resolved into another equal and parallel at any distance, and a pair whose plane is parallel to the plane of translation, and whose moment is the product of that force into the distance to which it is so transferred.

Thus, if a force $+P$ acts in the direction BA, (Fig. 28.) the effect will not be changed by applying to any point of the system, as D, two forces, each of them equal to P, and acting in the line EC parallel to BA, but in opposite directions, *i. e.* a force $+P$, acting at D in the direction DC, and a force $-P$, acting at the same point, in the direction DE. We have then three forces, viz. $+P$ applied at D, in a direction parallel to BA, and a pair of forces P, whose moment is $P.d.$; d being the distance to which the given force is transferred. The pair, thus generated, may afterwards be transferred or transformed, in any of the ways already specified.

A pair and a single force, not parallel to the plane of the pair, cannot be compounded into a single resultant.

Let the pair consist of the forces $+P$, $-P$, acting in the directions Aa, Bb; (Fig. 29.) and let $+P'$ be the single force, acting in the direction Cc, not parallel to the plane of the pair. If there were a single resultant R; a force $-R$, equal and opposite, would be in equilibrio with the force $+P'$, and the pair $+P$, $-P$. This supposed equilibrating force, $-R$, cannot be equal, parallel, and contrary to $+P'$: for then they would constitute a pair in a plane which, not being parallel, intersects the plane of the pair $+P$, $-P$; these two

pairs would then have a resultant pair, (Art. 2.) and therefore, the equilibrium would not subsist. This being observed, let Ff be the direction of $-$R, and let the force $+$P$'$ be transferred to o, a point in that line: by this translation, a pair will be generated; and we shall then have two pairs, and two single forces. The pairs are $+$P, $-$P, acting in the lines Aa, Bb, and $+$P$'$, $-$P$'$, acting in the lines Cc, og: and as the planes of these pairs intersect, it follows that they must give a resultant pair. The single forces are $-$R and $+$P$'$, acting in the directions oG, of, and these forces, which act at the same point, not being equal and opposite forces, must give a single resultant. We have then a pair and a single force, which cannot equilibrate. As the force P$'$, and the pair of forces P, cannot be equilibrated by a single force, it follows that they cannot be compounded into a single force.

Hence it follows, that two forces P, P$'$, not in the same plane, cannot have a single resultant.

For, let P$'$ be translated to a point in the direction of P, and there compounded with it into a resultant R: we have then a force R, and a pair of forces — P$'$, $+$P$'$, generated by the translation of $+$P$'$; and as P is inclined to the plane of this pair, R must also be inclined to the same plane; wherefore, the force R cannot be compounded with the pair into a single resultant.

Any number of forces, applied to the points of an invariable system, may be reduced to a single force and a pair.

For all the forces may be transferred to any one point of the system, and by these translations there will be generated so many pairs. All the forces so transferred may be compounded into a single force; and all the pairs into a single pair.

If the single forces applied at the same point are in equilibrio, there remains but the resulting pair: and if the pairs are in equilibrio, or if they give a resulting pair, whose plane contains the direction of the single resulting force, the whole

system of forces may be compounded into a single resultant. And if the translated forces are in equilibrio, and also the pairs, generated by the translations, the whole system of forces applied to the different points of the body is in equilibrio.

Conversely, if the system of forces applied to the different points is in equilibrio, the equilibrium must subsist separately, in the system of translated forces, and among the pairs generated by the translations, *i. e.* the former cannot have a resulting force, nor the latter a resulting pair: for were both to result, they could not equilibrate, and if either were to result exclusively, the system of forces from which it results could not be in equilibrio.

It remains to show how these conditions of equilibrium are expressed by the magnitudes of the forces, their directions, and points of application: and this is what is proposed in the following section.

SECTION IV.

CONDITIONS OF EQUILIBRIUM AMONG FORCES APPLIED TO DIFFERENT POINTS OF AN INVARIABLE SYSTEM, AND IN ANY DIRECTIONS.

1. It has been shown in last section, that the forces applied to an invariable system, whatever be their number, their intensities, their directions, or points of application, are always reducible to a single force, and a single pair: including the cases in which either or each of these resultants is equal to cypher. It was, moreover, shown that a single force and a single pair cannot equilibrate. Whence it follows, that for equilibrium among the forces applied, it is requisite that the resulting force, and the moment of the resulting pair, should be separately equal to cypher; and it is obvious, that when these conditions are fulfilled, the forces are in equilibrio.

To reduce these conditions to formulæ easily applied to particular cases, it would seem requisite to express these two resultants, as functions of the magnitudes of the forces, the coordinates of their points of application, and the angles by which their directions are determined: and then, to equate to cypher, each of these resultants so expressed. To make out these formulæ, and then to apply them to some important cases, is the business of the present section.

The process by which we arrive at these formulæ is as follows. Three axes being assumed, which meet at a solid angle; each of the forces is resolved into three, parallel to these axes; each component is then transferred to its parallel axis, and all in the same axis combined into one force,

equal to their sum. But by the translation of each component force to its parallel axis, there is generated a pair. Each of these pairs of forces is resolved into two, in the coordinate planes to which they are parallel: and all the pairs being thus brought into the three coordinate planes, those in each plane are combined into one pair.

In this manner all the forces applied to the system are reduced to three forces, directed along the axes, and three pairs in the three coordinate planes. The three forces thus obtained are readily compounded into one force, and the three pairs into one pair. But this further step is unnecessary for the present purpose, which is not to find the expression for the resultant force or resultant pair, but the expression of the conditions to be fulfilled, in order that each resultant should be equal to cypher. Now, the resultant of the forces directed according to the three axes cannot be equal to cypher, unless each of the three forces is, separately, equal to cypher: (Sect. I. Art. 12.) and that the same thing may be asserted of the moments of the pairs in the three coordinate planes, will appear in the same way from (Sect. III. Art. 2.) Wherefore, the two general conditions of equilibrium are equivalent to six; requiring that "the sum of the forces in each of the three axes should be equal to cypher, and that the sum of the moments of the pairs in each of the coordinate planes should be equal to cypher."

To illustrate this reasoning, let Ax, Ay, Az, be the three coordinate axes, (Fig. 30.) and let *os* represent one of the forces. Then if this line is made the diagonal of a parallelopiped, whose sides, *om*, *on*, *or*, are parallel to the three axes, the force *os*, may be resolved into three, represented in quantity and direction by these three lines. Any one of these, as *or*, may be transferred to its parallel axis Az, and by this translation there will be generated a pair of forces, *or*. Let then the line *ro*, produced if necessary, meet the plane yAx, at the point b; it may be supposed to be applied

at this point; the pair of forces $-or$, $+or$, are then applied on the line Ab: and drawing the lines bf, bg parallel to Ax, Ay, this pair may be resolved into two pairs of forces equal and parallel to the former, and applied on the lines Af, Ag. The force om, in the same way, produces an equal force in the axis Ax, and two pairs in the planes zAx, xAy: and the force on produces an equal force in the axis Ay; and two pairs in the planes zAy, xAy. All the forces being treated in the same way, if the sum of the forces in the axis Ax, the sum in Ay, and the sum in Az, are denoted by X, Y, Z, and the sums of the moments in the planes zAy, zAx, yAx, by L, M, N, we shall have for equilibrium

$$\mathrm{X} = 0. \qquad \mathrm{Y} = 0. \qquad \mathrm{Z} = 0. \qquad (1)$$

$$\mathrm{L} = 0. \qquad \mathrm{M} = 0. \qquad \mathrm{N} = 0. \qquad (2)$$

These expressions are now to be developed; and if the several forces in the axis Ax, are denoted by X′, X″, X‴, &c.; those in the axis Ay, by Y′, Y″, Y‴, &c.; those in the axis Az, by Z′, Z″, Z‴, &c.: the three first conditions are equivalent to the three following:

$$\left.\begin{array}{l} \mathrm{X}' + \mathrm{X}'' + \mathrm{X}''' + \text{\&c.} = 0. \\ \mathrm{Y}' + \mathrm{Y}'' + \mathrm{Y}''' + \text{\&c.} = 0. \\ \mathrm{Z}' + \mathrm{Z}'' + \mathrm{Z}''' + \text{\&c.} = 0. \end{array}\right\} \quad (3)$$

Of these forces, such as tend, by their directions, to increase the ordinates of their points of application within the same plane angle, may be deemed positive, and those which act in the opposite directions, negative. The sign $+$ with which they are connected in the formulæ, denotes that they are to be added with their proper signs.

Of the three equations which relate to the moments, it is to be observed, first, that the coordinates of the points of application of each of the forces, are those of three forces into which it is immediately resolved, and that when these last are transferred to the origin, and the pairs generated by

these translations resolved in the three coordinate planes, the same coordinates are the portions of the axes on which these component pairs are applied. Thus, Ag, Af, on which the two pairs of forces, *or*, are applied, are two of the coordinates of the point *o*, and so of the rest. Secondly, it is to be observed, that the directions of the forces of the pairs in each of the planes, being equally inclined to the axes on which they are applied, the product of the force of each pair into the distance measured on the axis on which it is applied, may be taken for the relative measure of its moment in the same coordinate plane. Wherefore, putting x', x'', x''', &c. for the coordinates of the points of application measured on the axis Ax; y', y'', y''', &c. for those measured on the axis Ay, and z', z'', z''', &c. for those measured on the axis Az, the moment of the pair of forces Z', in the plane zAx, is $Z'.x'$, and that of the pair of same forces in the plane zAy is $Z'.y'$; and so of the moments of all the pairs in the coordinate planes. Thirdly, the pairs in each of the coordinate planes are divided into two sets, distinguished by the axis, to which their forces are parallel. Thus, in the plane yAx there are the pairs whose moments are $Y'.x'$, $Y''.x''$, $Y'''.x'''$, &c., and the pairs whose moments are $X'.y'$, $X''.y''$, $X'''.y'''$, &c. and of these it is to be observed, that if the forces, and also the abscissæ, are positive in any one of the coordinate planes, the moments of these different sets are of contrary directions. Accordingly, the equations $L = 0$. $M = 0$. $N = 0$. are equivalent to the three following:

$$\left.\begin{array}{l}(Y'z'-Z'y')+(Y''z''-Z''y'')+(Y'''z'''-Z'''y''')+\&c.=0.\\(Z'x'-X'z')+(Z''x''-X''z'')+(Z'''z'''-X'''z''')+\&c.=0.\\(X'y'-Y'x')+(X''y''-Y''x'')+(X'''y'''-Y'''x''')+\&c.=0.\end{array}\right\}(4)$$

The negative signs in these formulæ denote, that the moments to which they are prefixed are to be subtracted with their proper signs.

The formulæ, as already presented, contain neither the forces immediately applied, nor the angles by which their directions are ascertained. But let the axes be rectangular, and let P', P'', P''', &c. be the forces, making with lines parallel to the axis Ax, the angles a', a'', a''', &c. with lines parallel to the axis Ay, the angles β', β'', β''', &c., and with lines parallel to the axis Az, the angles γ', γ'', γ''', &c. we shall have

$$X' = P'.\cos.a'. \qquad Y' = P'.\cos.\beta'. \qquad Z' = P'.\cos.\gamma'.$$

and so of the other component forces. Wherefore, making these substitutions in equations (3) and (4) they are presented as follows:

$$\left.\begin{array}{l} P'.\cos.a' + P''.\cos.a'' + P'''.\cos.a''' + \&c. = 0. \\ P'.\cos.\beta' + P''.\cos.\beta'' + P'''.\cos.\beta''' + \&c. = 0. \\ P'.\cos.\gamma' + P''.\cos.\gamma'' + P'''.\cos.\gamma''' + \&c. = 0. \end{array}\right\} (5)$$

$$\left.\begin{array}{l} P'(\cos.\beta'.z' - \cos.\gamma'.y') + P''(\cos.\beta''.z'' - \cos.\gamma''.y'') + \\ \quad P'''(\cos.\beta'''.z''' - \cos.\gamma'''.y''') + \&c. = 0. \\ P'(\cos.\gamma'.x' - \cos.a'.z') + P''(\cos.\gamma''.x'' - \cos.a''.z'') + \\ \quad P'''(\cos.\gamma'''.x''' - \cos.a'''.z''') + \&c. = 0. \\ P'(\cos.a'.y' - \cos.\beta'.x') + P''(\cos.a''.y'' - \cos.\beta''.x'') + \\ \quad P'''(\cos.a'''.y''' - \cos.\beta'''.x''') + \&c. = 0. \end{array}\right\} (6)$$

Such are the six conditions of equilibrium expressed by the magnitudes of the forces, the coordinates of their points of application, and the angles which determine their directions.

2. To find the geometrical magnitudes denoted by L, M, N, let the plane yAx pass through the point o, (Fig. 31.) and retaining the representations of the force P' and its components made in (Fig. 30.) let the lines mo, on, be produced to meet the axes Ay, Ax, at the points c and d. We shall then have

$$P'.\cos.a'.y' = X'.y' = om \times AC$$

and

$$\text{P}'.\cos.\beta'.x' = \text{Y}'.x' = on \times \text{A}d.$$

Whence,

$$\text{P}'.(\cos.a'.y' - \cos.\beta'.x') = om \times \text{A}c - on \times \text{A}d.$$

These products may be combined into one, by the following theorem:

"If from a point in the plane of a parallelogram, perpendiculars are let fall on the diagonal and contiguous sides, the product of the diagonal into its perpendicular distance from that point, is equal to the sum or difference of the products, had by multiplying each of the sides into its perpendicular distance from the same point; according as the point is placed without or within the angle contained by the sides."

To prove this proposition, let *ombn* (Fig. 32.) be the parallelogram, and A the point from which are let fall on the diagonal and sides the perpendiculars Ag, Ac, Ad. Also, from the points m, b, n, let mr, bp, nq, be perpendicular to the line Ao. There will then be

$$ob \times \text{A}g = 2\,\text{triang.A}ob = \text{A}o \times bp.$$

$$om \times \text{A}c = 2\,\text{triang.A}om = \text{A}o \times mr.$$

$$on \times \text{A}d = 2\,\text{triang.A}on = \text{A}o \times nq.$$

But $bp = mr \pm nq$. Wherefore $\text{A}o \times bp = \text{A}o\,(mr \pm nq)$;

$$i.\,e.\ ob \times \text{A}g = om \times \text{A}c \pm on \times \text{A}d,$$

the positive sign being taken when the point A is placed without the angle of the parallelogram; and the negative sign when that point is placed within.

This theorem may be expressed as follows: "A line being drawn from the angle of a parallelogram and in its plane, the product of the diagonal into the sine of the angle which it makes with that line, is equal to the sum of the products of the sides into the sines of the angles which they

make with the same line, when that line falls without; and to the difference of those products, when that line falls within the angle of the parallelogram."

In the formulæ, the moments of the pairs generated by any force are expressed as if the coordinates were both positive, or both negative, *i. e.* as if the origin A were taken within the angle *nom*, or the angle vertically opposite: accordingly, referring to (Fig. 31.)

$$P'.(\cos.\alpha'.y' - \cos.\beta'.x') = om \times AC - on \times Ad = ob \times Ag.$$

The line *ob* is the projection of the line *os* on the plane yAx; wherefore, the moment of the pairs generated by the force P′, in the plane yAx, or round the axis Az, is the product of this force projected on the plane yAx, into the perpendicular from the origin on this projection.

This may be otherwise expressed, for if the lines A*o*, A*s*, are drawn from the origin to the extremities of *os*, by which the force is represented, (Fig. 31.) and if the triangle A*os* is projected on the plane yAx, it may be stated that the moment of the force P′ in the plane yAx, or round the axis Az, is represented by twice the area of the projected triangle.

The projection of a force on a plane, is the product of that force into the cosine of its inclination to that plane, *i. e.* into the sine of the angle which it makes with a line perpendicular to that plane. Wherefore, the projection of P′ on the plane zAy, is $P'.\sin.\alpha'$: its projection on the plane zAx, is $P'.\sin.\beta'$: and that on the plane yAx, is $P'.\sin.\gamma'$. Denoting the projections of P″, P‴, &c. in the same way, and putting p', p'', p''', &c. for the perpendiculars from the origin, on the forces projected on the plane zAy. Also, q', q'', q''', &c. for the perpendiculars on the same forces projected on the plane zAx; and r', r'', r''', &c. for the perpendiculars on the same forces projected on the plane yAx; the equations (6) may receive the following more simple forms:

$$\left.\begin{array}{l} \text{P}'.\sin.\alpha'.p' + \text{P}''.\sin.\alpha''.p'' + \text{P}'''.\sin.\alpha'''.p''' + \&c. = 0. \\ \text{P}'.\sin.\beta'.q' + \text{P}''.\sin.\beta''.q'' + \text{P}'''.\sin.\beta'''.q''' + \&c. = 0. \\ \text{P}'.\sin.\gamma'.r' + \text{P}''.\sin.\gamma''.r'' + \text{P}'''.\sin.\gamma'''.r''' + \&c. = 0. \end{array}\right\} (7)$$

In (Fig. 31.) Ag perpendicular to ob, is perpendicular to the plane of projection sob; it is then the shortest distance between Az and that plane, and therefore the shortest distance between the lines Az and os; which enables us to express the three last conditions, by stating that "the sum of the products of the shortest distances of the several forces from each axis, into the projections of the same forces on the plane of the two remaining axes, is cypher."

3. The equations of equilibrium are now readily found, on any particular supposition, by considering how the six general formulæ would be thereby affected.

Thus; if the conditions of equilibrium were required for a system of forces meeting at a point, this point may be taken for the origin, and then the coordinates of the points of application being all cypher, the three equations (4), or the three equivalent equations, (6), would be necessarily fulfilled, and the perpendiculars from the origin on the directions of the forces being cypher, the same is true of equations (7). There remain therefore only the three first conditions, expressed by the three equations (1), or the equations (3) or (5), which are equivalent.

If the forces are parallel, by taking one of the axes, as Az, parallel to the forces, we have

$$\text{X} = 0. \qquad \text{Y} = 0.$$

and therefore the two first and the last of the six general conditions are necessarily fulfilled. Accordingly, the only conditions of equilibrium required in this case are those expressed by the last of the equations (1), (3), or (5), and the two first of the equations (2), (4), (6), or (7), and in this case, those equations become more simple; the equations (4) becoming

$$\text{Z}'.y' + \text{Z}''.y'' + \text{Z}'''.y''' + \&\text{c.} = 0.$$

$$\text{Z}'.x' + \text{Z}''.x'' + \text{Z}''.x'' + \&\text{c.} = 0.$$

and the corresponding equations (6) and (7) then assuming similar forms.

These conditions of equilibrium for parallel forces are expressed by saying, that "the sum of the forces should be equal to cypher, and that the sum of the moments round each of the axes in the perpendicular plane should be equal to cypher."

If the directions of the forces are in the same plane, this may be taken for one of the coordinate planes, as for example, for the plane $y\text{A}x$. Then $\text{Z} = 0. \cos.\gamma'$, &c. $= 0$, which reduces the equations (1), (3), (5) to the two first. And z', &c. being cypher, $\cos.\gamma'$, &c. being cypher, and the perpendiculars p', &c. q', &c. being also cypher, the three equations (2), (4), (6), (7), are reduced to the last. Moreover, $\gamma', \gamma'', \gamma'''$, &c. being right angles, the last of the equations, (7), assumes the form,

$$\text{P}'.r' + \text{P}''.r'' + \text{P}'''.r''' + \&\text{c.} = 0.$$

Wherefore, the three conditions of equilibrium, for forces whose directions are in the same plane, are expressed by saying, that "the sums of the forces resolved in directions parallel to two axes in that plane, should be separately equal to cypher, and that the sum of the moments round the third axis should also be equal to cypher."

If the directions of the forces are parallel and in the same plane, that plane may be taken for one of the coordinate planes, as for example, for the plane of the axes of x and y; also, one of these axes, as the axis of x, may be taken parallel to the directions of the forces; and then it will appear, in the same way, that the conditions are reduced to the first and last, which require "that the sum of the forces parallel to $\text{A}x$, should be cypher, and the sum of the moments round the axis $\text{A}z$ equal to cypher."

4. When the six conditions of equilibrium are not satisfied there must be a single resultant, or a resulting pair, or both.

To have a resulting pair without a resulting force, it is requisite, first, that the partial moments L, M, N, should not all be equal to cypher: for if each of these moments is cypher, there is no resulting pair. Secondly, the partial resultant forces must be, each, equal to cypher, *i. e.* we must have

$$X = 0. \qquad Y = 0. \qquad Z = 0.$$

otherwise these forces would have a resultant, and it has been shown, that a force and a pair cannot be compounded into a pair.

To have a resulting force without a resulting pair, it might, at first view, appear requisite, that besides having some one of the quantities X, Y, z, different from cypher, we should have

$$L = 0. \qquad M = 0. \qquad N = 0.$$

But it is to be considered, that though the forces of the system are reducible to a single resulting force, yet if that resultant does not pass through the origin, it generates a pair when transferred to that point: and that it is in this complex form it would be at first presented by the process for composition, above described: whence it appears, that the last equations express the conditions requisite for a single resultant passing through the origin; and that, generally, the case of a single resultant does not require this condition.

To find the conditions for a single resultant, it is therefore to be supposed, that the moments L, M, N, may be different from cypher; and it is to be inquired, whether by any change of the origin, they may be all reduced to cypher. By assuming any other point whose three coordinates are x, y, z, and taking this point for the origin, the new coordinates would be diminished by those quantities: whereby the

sum of the moments of the pairs of forces Y', Y'', Y''', &c. in the plane zy, would become

$$Y'.z'+Y''.z''+Y'''.z'''+\&c.-(Y'+Y''+Y'''+\&c.)z.$$

that is,

$$Y'.z'+Y''.z''+Y'''.z'''+\&c.-Y.z.$$

Those of the forces Z', Z'', Z''', &c. in the same plane would become

$$Z'.y'+Z''.y''+Z'''.y'''+\&c.-Z.y.$$

Making the corresponding changes in the moments in each of the other planes, the three last of the six general conditions of equilibrium will become

$$\left.\begin{aligned} L-Yz+Zy&=0.\\ M-Zx+Xz&=0.\\ N-Xy+Yx&=0.\end{aligned}\right\} \qquad (8)$$

From these three equations, any two of the three indeterminate quantities being exterminated, there results the equation

$$XL+YM+ZN=0. \qquad (9)$$

This equation expresses the relation, which should subsist, among the three partial resultant forces and the three partial resultant moments, in order that the forces of the system should have a single resultant.

This condition may be otherwise investigated, from the principle, that the resultant of the forces transferred to the origin, can be compounded with the resultant moment whenever it is parallel to the plane of that moment; and that in no other case is it possible that they should be compounded into a single resultant. Therefore, the condition inquired after is the same as that to be fulfilled, in order that the direction of the resulting force should be parallel to the plane of the resulting moment, or, which is the same thing, perpendicular to the axis of that moment.

Let R denote the resultant of the forces transferred to the origin; α, β, γ, the angles which it makes with the axes Ax, Ay, Az, we shall have

$$\cos.\alpha = \frac{X}{R}. \qquad \cos.\beta = \frac{Y}{R}. \qquad \cos.\gamma = \frac{Z}{R}.$$

Putting G for the resulting moment, and λ, μ, ν, for the angles made by its axis with the coordinate axes Ax, Ay, Az, we have

$$\cos.\lambda = \frac{L}{G}. \qquad \cos.\mu = \frac{M}{G}. \qquad \cos.\nu = \frac{N}{G}.$$

Now in order that R should be perpendicular to the axis of the moment G, the cosine of the angle contained by those lines must be cypher, *i. e.*

$$\cos.\alpha\ \cos.\lambda + \cos.\beta.\ \cos.\mu + \cos.\gamma.\ \cos.\nu = 0.$$

which, putting for the cosines their values as above, gives the equation

$$XL + YM + ZN = 0.$$

When this condition is not fulfilled, there is a resulting force and a resulting moment: and this completes the account of the conditions required; first, for equilibrium, when there is neither resulting force nor resulting moment; secondly, for a resulting moment; thirdly, for a single resulting force; and fourthly, for both a resulting force and a resulting moment.

The values of the resulting moment, and of the angles made by its axis with the three coordinate axes, have been already given: likewise, the value of the resultant force, and of the angles made by its direction with the same axes, when the direction passes through the origin; and these values are not changed when it is compounded with a pair to whose plane it is parallel. But by this composition, the line of direction is changed for another to which it is parallel, in a plane parallel to the plane of the pair; and the distance to

which it is tranferred is $\frac{G}{R}$. Making the directions of the forces of G parallel to that of R, the side on which R is thus transferred, is that of the force of G, which corresponds with it in direction, as compared to the force of the contrary direction. (Sect. III. Art. 3.)

If, however, a formula is required for the point of application of the single resultant, this is readily supplied, for as the point of application of a force may be any point, taken indifferently in its line of direction, x, y, z, are the coordinates of any point in the direction of this resultant; and, therefore, equations (8), which express the relations among these coordinates, are the equations of this line. These, constituting but two independent equations, do not give the absolute, but the relative values of the coordinates. This, however, is sufficient to furnish the values of two of the coordinates, for any supposition made with respect to the magnitude of the third. Thus, if it were proposed to find the intersection of the resultant with the plane of zx: by making $y=0$, in the first and third of these equations, we have this point determined by the equations,

$$z = +\frac{L}{Y}. \qquad x = -\frac{N}{Y}.$$

And in the same way, by making $z=0$, in the first and second of those equations, we have

$$x = +\frac{M}{Z}. \qquad y = -\frac{Z}{Z}.$$

for the intersection with the plane of yx.

It was observed of equation (9), that it expresses the condition to be satisfied, in order that the forces of the system should have a single resultant; but it does not follow that there will be a single resultant whenever this condition is satisfied: for it must be remembered, that the condition was investigated on the supposition of the existence of R, the

resultant of the forces transferred to the origin, *i. e.* on the suppositon that X, Y, Z, were not, each of them, equal to cypher. If each of these is cypher, there will be only a resulting pair, and equation (9) is equally satisfied.

To apply this to the cases of parallel forces, and of forces in the same plane; if the forces are parallel, and their sum different from cypher, they have a single resultant, and by taking one of the axes, as for example, the axis Ax, parallel to their directions, we have

$$\mathrm{Y}=0. \qquad \mathrm{Z}=0. \qquad \mathrm{L}=0.$$

which reduces the equations (8) to

$$\mathrm{M}+\mathrm{X}z=0.$$
$$\mathrm{N}-\mathrm{X}y=0.$$

giving

$$y=+\frac{\mathrm{N}}{\mathrm{X}}. \qquad z=-\frac{\mathrm{M}}{\mathrm{X}}.$$

both constant quantities.

If the forces act in the same plane, that plane may be taken for the plane of yx; and then

$$\mathrm{Z}=0. \qquad \mathrm{L}=0. \qquad \mathrm{M}=0.$$

which reduces the equations (8) to

$$\mathrm{Y}z = 0.$$
$$\mathrm{X}z = 0.$$
$$\mathrm{N}-\mathrm{X}y+\mathrm{Y}x=0.$$

the last giving

$$y=\frac{\mathrm{N}+\mathrm{Y}x.}{\mathrm{X}}.$$

If the forces are parallel and in the same plane, they may be supposed to be parallel to Ax, and in the plane of xy; and then

$$\mathrm{Y}=0. \qquad \mathrm{Z}=0. \qquad \mathrm{L}=0. \qquad \mathrm{M}=0.$$

which reduces the equations (8) to

$$\begin{aligned} \text{X}z &= 0. \\ \text{N} - \text{X}y &= 0. \end{aligned}$$

giving

$$y = + \frac{\text{N}}{\text{X}};$$

a constant quantity.

To recapitulate the principal truths established in this article:

When each of the six quantities, X, Y, Z, L, M, N, is cypher, the system of forces is in equilibrio.

When each of the three quantities X, Y, Z, is cypher, some one or more of the quantities L, M, N, being different from cypher, the system is reducible to a pair without a single force.

When each of the quantities L, M, N, is cypher, some one or more of the quantities X, Y, Z, being different from cypher, the system of forces is reducible to a single force passing through the origin.

Some one or more of the partial resulting forces, and also, some one or more of the partial resulting moments being different from cypher, if the condition expressed by equation (9) is satisfied, there is a single resulting force, which does not pass through the origin.

Finally, retaining the former part of the last supposition respecting the existence of a partial resulting force, and a partial resulting moment, if the condition expressed by equation (9) is not satisfied, there is a resulting pair, and also a resulting force, and the latter not being in the plane of the former, the system cannot be reduced to a single force, or to a single pair.

SECTION V.

OF THE CONDITIONS OF EQUILIBRIUM, WHEN THE BODY IS IN PART RESTRAINED BY FIXED OBSTACLES; AND OF THE PRESSURES ON THE POINTS OF CONTACT.

1. WHEN a body is found not to be at liberty to move, or when its motion is not in the direction of the force impressed, it is plain that some new force is introduced, which in the one case, is in equilibrio with the force actually applied, and which in the other case, compounded with that force, gives a resultant in the direction of the motion produced. In the former case, the force introduced is, plainly, equal and opposite to that applied; and in every case, it were easy to find, experimentally, the magnitude and direction of the new force: for if a third force is applied sufficient to maintain the body in a state of rest, it is evident that the three forces are in equilibrio; and of these, two are given in magnitude and direction; whereby the force sought is fully ascertained. But this experimental investigation is wholly unnecessary for the present purpose.

If a force is applied to a fixed point, it is destroyed; or, more properly speaking, there is another force brought into action, with which it is in equilibrio.

When a force is applied to a material point, whose movements are confined to a certain line; if the direction of the force is perpendicular to that line, there is no reason why it should move the point along that line, in one direction, rather than in the opposite direction; whence we may conclude that no motion will ensue. But if the force is oblique, it may be resolved into two; of which, one is perpendicular

and the other parallel to the line: the former shall be wholly counteracted, but the latter shall produce its full effect.

When a force is applied to a material point in contact with a plane surface; if the force is directed in the perpendicular and towards the surface, it cannot move the point from the plane towards which it is directed; neither can it move the point through the plane considered as impassable; and acting perpendicularly, there is no reason why it should move the point, on the surface of the plane, in any one direction rather than another. Hence it may be inferred, that the perpendicular force cannot produce any motion in the point; and, therefore, that it is effectually counteracted.

But, if the direction of the force is oblique to the plane, it may be resolved into two components, one of them perpendicular to the plane, and the other in that plane: of these, the former is wholly counteracted, but the latter, which is not obstructed, shall produce its full effect in driving the point along the plane. The same things hold when the surface is a curve of any kind, inasmuch as the surface at the point of application may be taken for that of its tangent plane.

Hence we derive the following conclusions relative to the force thus brought into action.

The force thus excited, being never exerted but in opposition to some force actually applied, and never exceeding the measure of this force, is merely a counteracting force; for which reason it is denominated a force of resistance.

With respect to the direction of this force of resistance, it appears that a fixed point is capable of a resistance in any line directed from that point. That the resistance of a line is directed perpendicularly from that line; and, therefore, that the resistance, at any point of the line, is confined to a plane perpendicular to the line at that point: and that the

resistance of a surface is directed from the surface in the perpendicular at the point of contact.

In the two last cases, it is supposed, that the force is not immediately applied to a point in the line or plane, for then it would be applied to a fixed point, and would belong to the first case; but that it is applied to a material point, whose movements are restricted by the line or plane.

The magnitude of the power of resistance in such obstacles, *i. e.* the magnitude of the force of resistance which they are capable of exerting, is unlimited; this force being always equal to that by which it is excited, *i. e.* to the force applied, resolved in a direction opposite to that of the resistance.

The force which is equilibrated by the resistance is denominated a pressure.

To seek the conditions of equilibrium, in the case of a body restrained by fixed obstacles, is to seek the conditions to be satisfied, in order that the forces actually applied, may be equilibrated by the resistances: and here it is evident, that as the resisting forces may be of any magnitudes, the inquiry is limited to the conditions to be satisfied, in order that the forces immediately applied, or their resultants, should be opposed in direction to the resistances. In other words, the inquiry is, what forces of the system are equilibrated by the resistances? and then, the conditions are those to be satisfied, in order to insure the equilibrium of the remainder. And as it is of the nature of all resistances to produce either a total or partial equilibrium, it may be expected that the general conditions of equilibrium, when not altogether satisfied by the resistances, shall leave those to be otherwise provided for, fewer, and in form more simple than for a body altogether free.

2. If the system to which the forces are applied contains a fixed point, it is requisite and it is sufficient for equili-

brium, that the forces should be reducible to a single resulting force passing through the fixed point.

But when the forces are referred to three axes meeting in a solid angle, the condition required, in order that they should have none but a single resultant passing through the origin, is that expressed by the equation $G = 0$. which imports, that the forces should have no resulting moment relative to the origin. Accordingly, "where there is a fixed point, it is requisite and it is sufficient for equilibrium, that the forces should have no resulting moment relative to that point."

The condition $G = 0$. was shown to be equivalent to these three,

$$L = 0. \qquad M = 0. \qquad N = 0.$$

Accordingly, when there is a fixed point, it is requisite for equilibrium, that "the forces applied should have no resulting moment round any one of three axes meeting at the fixed point in a solid angle; and these conditions being satisfied, the equilibrium is established."

When the forces of the system are parallel, one of the axes, as for example, the axis of x, may be taken parallel to the directions of the forces; and then, of necessity, the value of L is cypher, which reduces the equations of condition to

$$M = 0. \qquad N = 0.$$

importing, that the equilibrium is established, when there is no resulting moment about either of the two axes, whose plane is perpendicular to the directions of the forces.

When all the forces are directed in a plane, containing the fixed point, that plane may be taken for the plane xy; there is in that case no moment relative to the axis of x, or the axis of y; *i. e.* of necessity, we have $L = 0$. $M = 0$. wherefore,

$$N = 0.$$

is the only condition to be provided for: *i. e.* when the forces act in a plane containing a fixed point, there will be equilibrium, when there is no moment about an axis perpendicular to that plane.

The resistance made by the fixed point, being in equilibrio with the forces of the system transferred to that point, in parallel directions, it follows that the pressure is the resultant of those forces so transferred.

4. If the forces are applied to a body which contains two fixed points, the line by which those points are connected, and therefore, every point in that line is fixed. Each point of this line is capable of resisting in every direction: but it is evident, that among these forces of resistance there can be no pairs of parallel forces, except those whose planes contain the fixed line.

To investigate the conditions of equilibrium with such resistances, let one of the fixed points be made the origin, and the connecting line the axis of x; and let the three axes be rectangular. Then, the forces being reduced in the usual way, to three forces acting in the axes themselves, denoted by X, Y, Z, and three pairs in the three coordinate planes, whose moments are denoted by L, M, N; the forces X, Y, Z, being applied to a fixed point, are effectually counteracted.

The planes of the moments M, N, contain the fixed line, and therefore the forces of these moments may be applied to that line; whereby those forces are also effectually resisted. It is otherwise evident, that these moments must be equilibrated by moments of resisting forces, inasmuch as by containing the fixed line in their planes, the motion of rotation, which each is fitted to produce in its own plane, is effectually resisted. There remains then only the moment L, and this cannot be equilibrated but by a moment whose plane is perpendicular to the axis of x; *i. e.* to the fixed line. But there can be no pair of resisting forces, except in a plane which contains this line; wherefore, this moment cannot be

equilibrated by the resistances. Accordingly, the condition expressed by the equation $L = 0$. remains to be satisfied independently of the forces of resistance; all the other conditions being necessarily satisfied by those forces. Wherefore, "for equilibrium in the case of two fixed points, it is requisite and sufficient, that there should be no moment of rotation round the line by which those points are connected."

If the body is at liberty to slide along the axis, this axis will oppose no resistances, except to the forces to whose directions it is perpendicular; but such are all the forces, equilibrated by the resistances in the former case, except the force X. Therefore, to the condition $L = 0$, required in the former case, there is now added the condition $X = 0$, *i. e.* "for equilibrium when the body is at liberty to slide along a fixed axis, it is requisite that there should be no resulting force in the direction of that axis, and no moment of rotation about it."

5. Hitherto the line, containing two fixed points, was considered generally as a fixed line; and this was fully sufficient when the conditions of equilibrium were sought. But if the line becomes fixed by the securities afforded to two points of that line, it is evident that its powers of resistance are supplied from those securities, and that by them, the pressures on the various points of the line must be sustained; and it is desirable to ascertain the pressure sustained by each; or as the problem is usually stated, to determine the pressures on each of two fixed points.

If the forces applied consist of a pair, whose plane contains the two fixed points, the pair may be turned round in its own plane, so that its forces shall be perpendicular to the line which connects those points. It may then be transformed into another, of equal moment, having the distance between the fixed points for that between the directions of its forces, and these forces may be then applied at those

points. From which it appears, that the pressure on each of the points is the moment itself, divided by the distance between the fixed points, *i. e.* putting G for the moment, a for the distance between the fixed points, and p for the pressure on either, we have

$$p = \frac{G}{a}.$$

The pressures at the two points are both in the plane of the moment, and both perpendicular to the line which connects the fixed points; but in contrary directions with respect to that line.

If the line connecting the fixed points is parallel to the plane of the pair, the measure of the pressures is the same, inasmuch as the pair may be transferred to the parallel plane in which that line is contained.

But if the line connecting the fixed points is inclined to the pair in the angle a, the pair being resolved in two planes, of which, one contains the line, and the other is perpendicular to the same line; the moments of these two components are

$$G . \cos. a. \qquad G . \sin. a.$$

of these, the latter not being equilibrated, either in the whole or in part, by the resistances, makes no charge on the fixed points, and the pressure made on each of them by the former, is

$$\frac{G . \cos. a}{a}.$$

To find the pressures resulting from any system of forces on two fixed points, is to find the pressures made by the forces X, Y, Z, and by the moments L, M, N.

Treating the forces as before, it is evident that the pressure sustained at the origin, in the direction of each axis, is due to the partial resultant directed along that axis, and to

the moments M and N; it being already shown that the moment L could not charge any point of the line to which its plane is perpendicular. The pressures made by these moments, at the origin, are,

$$-\frac{\text{M}}{a},$$

in the direction of the axis of z,

and $$+\frac{\text{N}}{a},$$

in the direction of the axis of y; and at the other fixed point the same, but in the contrary directions. Wherefore, the pressures at the origin are expressed as follows:

In the direction of the axis of z,

$$\text{Z}-\frac{\text{M}}{a}=\text{Z}+\frac{\text{X}z-\text{Z}x}{a}=\frac{\text{Z}(a-x)+\text{X}z}{a}.$$

and in the direction of the axis of y,

$$\text{Y}+\frac{\text{N}}{a}=\text{Y}+\frac{\text{Y}x-\text{X}y}{a}=\frac{\text{Y}(a+x)-\text{X}y}{a}.$$

The pressures at the other fixed point are these:

Parallel to the axis of z,

$$+\frac{\text{M}}{a}=\frac{\text{Z}x-\text{X}z}{a}.$$

parallel to the axis of y,

$$-\frac{\text{N}}{a}=\frac{\text{X}y-\text{Y}x}{a}.$$

We should arrive at the same conclusion by considering that the force Z is transferred from the origin, by the moment M, to a distance $\frac{\text{M}}{\text{Z}}$. And the force Y, by the moment N, to the distance $-\frac{\text{N}}{\text{Y}}$. For the pressures made by the forces Z and Y, transferred to these distances, would be

$$\frac{Z\left(a-\frac{M}{Z}\right)}{a}=\frac{Za-M}{a}. \qquad \frac{Y\left(a+\frac{N}{Y}\right)}{a}=\frac{Ya+N}{a}.$$

at the origin.

And $$+\frac{M}{a}. \qquad -\frac{N}{a}.$$

at the other fixed point.

Putting p and q for the pressures at the origin, in the directions of the axes of z and y; and p' and q' for those at the other fixed point, in the parallel directions; and putting P, P′ for the resultants of these pressures at each point, we have

$$P=\sqrt{(p^2+q^2)}$$

for the pressure at the origin;

$$P'=\sqrt{(p'^2+q'^2)}$$

for that at the other fixed point.

Moreover, putting γ, for the angle made by P, with the axis of z; and γ', for that made by P′, with a line parallel to the same axis, we have

$$\cos.\gamma=\frac{p}{\sqrt{(p^2+q^2)}}. \qquad \cos.\gamma'=\frac{p'}{\sqrt{(p'^2+q'^2)}}.$$

Thus, these pressures are completely determined in magnitude and direction.

But this does not relate to the pressure made by the force X; neither is it possible to ascertain the manner in which this force is divided between the fixed points; for as every force may be applied indifferently at any point in its line of direction, it is evident that the force X may be applied at either of the fixed points, or divided between them in any ratio whatsoever. All that we can know concerning the pressures produced by it is, that the sum of those pressures is equal to the force by which they are made, and the same thing is true of the pressures parallel to either of the other

axes, *i. e.* their sum is equal to the partial resultant of all the forces in the parallel axis, as will immediately appear by referring to the values of those pressures given above.

6. When there are three fixed points in the system, and not in the same right line, the whole system becomes immoveable, and therefore, every point in it is to be regarded as a fixed point. It is evident, therefore, that the equilibrium is, in this case, necessarily established by the resistances; the forces, whatever their directions, being applied at fixed points.

But as the system may be rendered immoveable, by securing the positions of three points not in the same right line, it is evident that the pressures, produced by the forces of the system, must be ultimately sustained by those securities, and therefore it may be expected that it should be shown, how those pressures are distributed upon three such points, and how their values and directions may be ascertained.

7. To find the pressures made by any system of forces on three fixed points not in the same right line: let one of the fixed points, a, be taken, as before, for the origin; (Fig. 33.) and the lines ab, ac for the axes of x and y; also, the axis of z perpendicular to the plane xy. Then, reducing the whole system of forces to three single forces, X, Y, Z, acting in the three axes, and three moments, L, M, N, in the three coordinate planes, the forces X, Y, Z, and the moments M, N, may be distributed between the points a and b, as before, when these were the only fixed points; observing, that the moment N, when designated by the forces and the ordinates of their points of application, should be multiplied into the sine of the angle bac: thus,

$$\mathrm{N} = (\mathrm{X}y - \mathrm{Y}x)\ \text{sin. ang. } bac.$$

Putting e for the line ac, the moment L produces at a the pressure $+\frac{\mathrm{L}}{e}$, in the direction of the axis of z; and the equal

and contrary pressure at the point *c*. In this way, we may arrive at expressions for the pressures at the several points; but this, in general, would be useless, as it is evident that the distribution of the pressures is arbitrary: thus, the force X may be indifferently applied at *a* or *b*, or divided between them in any ratio; and the same is manifest respecting the force Y, and the points *a* and *c*. Also, the moment N may be indifferently applied on any one of the lines *ab*, *ac*, or *bc*. It is only when the forces X, Y, and the moment N are cypher, *i. e.* it is only when there is no resulting force, except in the perpendicular to the plane of the triangle, and no resulting moment, whose plane is not perpendicular to the same plane, that the question, relative to their bearings on the three angular points, admits of a determinate answer. For a force is equilibrated by the resistance of a single point in its direction, and therefore, the addition of a second fixed point in that direction, renders the problem indeterminate. Also, a moment is equilibrated by the resistances of two fixed points in its plane, and therefore the problem is rendered indeterminate by a third fixed point in that plane.

The problem being determinate only when the direction of the resulting force, or the plane of the resulting pair is perpendicular to the plane of the triangle made by joining the three fixed points, it may be proper to show how to find the pressures made on those points, by such forces or pairs.

For the pressures made by a force Z, whose direction is perpendicular to the plane of the triangle *abc*: let *o* be the point where the plane is met by the direction of this force; and from one of the three angles, as *a*, let the line *ao* be drawn, meeting the opposite side, as at *d*; the force Z may be distributed on the points *a* and *d*: the portion sustained by the point *a*, being

$$Z \times \frac{od}{ad}.$$

and that by the point d, being

$$z \times \frac{ao}{ad}.$$

This last may be distributed on the points b and c; that on the point b, being

$$z \times \frac{ao}{ad} \times \frac{dc}{bc}.$$

and that on the point c, being

$$z \times \frac{ao}{ad} \times \frac{db}{bc}.$$

If the sum of the pressures, or z, is represented by the area of the triangle abc, the portion sustained by the point a shall be represented by the area of the triangle boc: and the sum of the pressures on b and c, by the four-sided figure $aboc$. Moreover, this sum is distributed on those points, in the ratio of dc to db; *i. e.* in the ratio of the area of the triangle aoc to the area of the triangle aob. Whence it follows, that if from the point of the plane where it is met by the direction of the force z, three lines are drawn to the angles of the triangle, the force z being represented by the area of this triangle, those at the three points shall be represented by the areas of the triangles, into which it is so divided, that at each angle, by the area of the triangle placed on the opposite side.

The same theorem will be true, though the point o should fall without the area of the triangle abc; the pressure on the point d being then in a direction corresponding to that of the force z, and the pressure on a in a direction parallel but contrary.

The pressures made by a pair on three fixed points are ascertained as follows: the plane of the pair being perpendicular to the plane of the triangle, made by joining the fixed points.

Let the pair be resolved into two pairs, whose planes shall contain two sides of the triangle. The pressures made by each component pair are had by dividing its moment by the line on which it is applied.

Or thus: from one of the angles, as a, let the line ad be drawn to the opposite side, parallel to the plane of the pair. (Fig. 33.) The pressures may be supposed, in the first instance, to be made on a and d; and putting G for the moment of the pair, as before, the pressure at a, and also that at d, is

$$\frac{G}{ad}.$$

The pressure at d is supported at b and c, and the parts supported at those points are expressed by the fractions $\frac{dc}{bc}$, and $\frac{db}{bc}$. Wherefore, the pressure at b will be

$$\frac{G}{ad} \times \frac{dc}{bc}.$$

and that at c,

$$\frac{G}{ad} \times \frac{db}{bc}.$$

If the line ad is bisected at o, and if from this point lines are drawn to the angles of the triangle, the pressures at the angles shall be proportional to the three triangular areas; the pressure at each angle being as the triangle placed on the opposite side.

For if the sum of the pressures, taken without regard to their signs, is $\frac{2G}{ad}$, and if this is represented in magnitude by the entire triangle abc, its half, which is supported at a, may be represented by the area of the triangle boc, and then the four-sided figure $baco$, shall represent the sum of the pressures on the points b and c, and this is divided between

them in the ratio of dc to db; *i. e.* of the triangle coa to the triangle boa.

If the point d lies between the points b and c, it is evident that the pressures at a and d shall be of contrary directions; and that, in this case, the pressures at b and c correspond in direction with that at d. But if d falls without the triangle, the point b, or the point c, beyond which it falls, shall sustain the pressure equal and contrary to the sum of those sustained by the two remaining points.

If the directions of the resultant forces, or the planes of the resulting pairs, are not perpendicular to the plane of the triangle, they may be resolved in the plane, and in the perpendicular to the plane; the pressures made by the forces and pairs acting in the plane are indeterminate, but those made in the perpendicular directions are fully determined.

If the body touches a plane in a single point, any force of resistance which may be excited at that point, is perpendicular to the plane. Therefore, when the forces of the system are not in equilibrio, they may equilibrate with the resistance, provided they are reducible to a single resultant, directed perpendicularly towards the plane at the point of contact. To see how these conditions may be expressed by means of the quantities which enter into the general formulæ, let the point of contact be taken for the origin, and the plane for that of xy, and the axes rectangular. Then, in order that there should be a single resultant passing through the origin, it is requisite that each of the moments L, M, N, should be cypher. And in order that this same force should be perpendicular to the plane, it is requisite that each of the partial resultants X, Y, should be cypher. Wherefore, for equilibrium, it is requisite that each of the five following equations should be satisfied,

$$\text{X} = 0. \qquad \text{Y} = 0. \qquad \text{L} = 0. \qquad \text{M} = 0. \qquad \text{N} = 0.$$

and these being satisfied by the forces of the system, there

will be a single resultant, which passes through the origin in a line perpendicular to the plane.

But besides the conditions expressed by the five equations given above, it is further required, that the resulting force should be directed towards the plane. For if it is directed from the plane, no resisting force is excited; and then, the six conditions are to be satisfied, as for a body perfectly free.

The pressure on the point of support is the force Z; which is the sum of the forces resolved in directions perpendicular to the plane, and transferred to the origin in parallel directions. When the equilibrium is established with the resistances, this is the resultant of the forces of the system.

If the body touches the plane in two points, the only forces of resistance which can be excited, are two such forces, perpendicular to the plane at the points of contact: and as these are parallel forces, of corresponding directions with respect to the line which joins the points of application, they have a parallel resultant, applied at some intermediate point. In order, therefore, that the equilibrium may be established by means of such resistances, it is requisite that the forces of the system should be reducible to a single resultant, directed perpendicularly towards the plane, and to some point intermediate between the points of contact. And when these conditions are satisfied, the forces of the system will be necessarily equilibrated by the resistances.

To see how these conditions are expressed by means of the quantities which enter into the general formulæ, let one of the points be taken for the origin, the connecting line for the axis of x, and the plane for that of xy. Then, in order that there may be a single resultant perpendicular to the plane of xy, it is requisite, in the first place, that each of the quantities X and Y should be cypher; for if either of these

is different from cypher, R is oblique to the plane of xy: then, if R can be compounded with G, the general resulting moment, into a single resultant, this, which is the general resultant of the system, is parallel to R, and, therefore, oblique to the plane of xy: and if R cannot be so compounded with G, the forces of the system are not reducible to a single resultant. Whence it follows, that unless each of the partial resultants, X and Y, is cypher, the forces of the system are not reducible to a single resultant perpendicular to the plane of xy. Accordingly, the equations

$$X = 0. \qquad Y = 0.$$

are to be satisfied. Z is the only partial resultant force that now remains. This, being in the planes of the moments L and M, may be compounded with both of these moments into a single force, equal and parallel to Z, and, therefore, perpendicular to the plane xy. But a force parallel to Z, and therefore perpendicular to the plane of N, cannot be compounded with this moment into a single resulting force; whence it follows, that the equation $N = 0$. is also to be satisfied.

Further; M may be compounded with Z, and the resultant is a single force equal and parallel to Z, in the plane of zx; and this resultant of Z and M may be compounded with the moment L into a single force, which will be also equal and parallel to Z: but this resultant of Z, M, and L, is not in the plane of zx; whence it follows, that the equation $L = 0$. is also to be satisfied. Wherefore, the equations to be satisfied, are

$$X = 0. \qquad Y = 0. \qquad L = 0. \qquad N = 0.$$

The moment of M may remain, but under certain restrictions. For as it is requisite that the single resultant of the system should be applied to the line connecting the points of support, at a point not beyond those limits; it follows, that the moment M should not exceed the product of Z into the

distance between the points of support, *i. e.* putting a for that distance, M should not exceed Za. For M compounds with Z a force equal and parallel, in the plane of zx; and applied at a point of the axis of x, whose distance from the origin is $x = \frac{\text{M}}{\text{Z}}$. Wherefore, if M exceeds Za, x shall be greater than a, *i. e.* the final resultant of the system shall be applied to the line connecting the points of support, at a point in that line produced beyond them.

It is to be further observed of this moment, that if it is applied on the axis of x, the force which corresponds in direction with Z, must be that which has the greater positive or the lesser negative abscissa. For, were it the contrary, the moment M, however small, when compounded with Z, would remove its point of application, out of the origin, on the side of the negative abscissæ, *i. e.* beyond the limits prescribed for the point of application of the final resultant, in the case of equilibrium.

The magnitude of this final resultant is Z, the sum of the forces of the system resolved in directions parallel to the axis of z, *i. e.* perpendicular to the plane of support; and the abscissa of its point of application on the axis of x, is given by the equation $x = \frac{\text{M}}{\text{Z}}$.

The pressures on the points of support are now immediately found. For putting p for the pressure at the origin, and q for that at the other point of support, we have

$$p : \text{Z} : a - x : a. \quad \text{giving } p = \frac{\text{Z}.(a-x)}{a}.$$

$$q : \text{Z} : x : a. \qquad \text{giving } q = \frac{\text{Z}.x}{a}.$$

i. e. putting for x its value,

$$p = \frac{\text{Z}a - \text{M}}{a}. \qquad q = \frac{\text{M}}{a}.$$

If the body touches the plane in more than two points arranged in the same right line, all but the extreme points are useless for the establishment of equilibrium. For, if the forces of the system have a single resultant applied between any two of the points, *a fortiori*, it will be applied between the extreme points. Wherefore, the question relative to the conditions of equilibrium is to be treated as if the body touched only at the two extreme points. But in the estimate of the pressures on the points of support, a great change is made, as it becomes impossible to ascertain any thing more than the sum of the pressures, when there are more than two points of support, and these in the same right line regarded as inflexible. For certain pressures may be removed from two points to an intermediate point; the pressures so removed, being reciprocally as the distances to which they are removed. And conversely, any pressure sustained by a point may be distributed upon two other points, between which it is placed, in portions, which are reciprocally as the distances of the point so exonerated, from those to which the pressure is transferred.

If the body touches the plane in three points not arranged in the same right line, there may be so many forces of resistance excited: and the direction of each of these forces of resistance is perpendicular to the plane. These constitute a system of parallel forces of corresponding directions: therefore they are reducible to a single force equal to their sum. The point of application of this resultant is in the area of the triangle made by joining the points of support. Hence it follows, that in order to equilibrate with the resistances, the forces of the system must be reducible to a single resultant perpendicular to the plane, and directed towards some point, not exterior to the triangle, made by joining the points of support. And when these conditions are satisfied, the equilibrium shall be necessarily established by the resistances: inasmuch as such a force can always

be distributed on the points of support, and in parallel directions.

To see how these conditions may be expressed, by means of the quantities which enter into the general formulæ, let one of the points of contact be taken for the origin; the line connecting it with a second point for the axis of x; the plane for that of xy; and the axes rectangular. Then, in order that the forces of the system may have a single resultant perpendicular to the plane, it may be proved, as in the last case, that X, Y, N, should be, severally, equal to cypher. There will then remain the partial resultant Z, and the two moments L, M. It is not required that either of these moments should be cypher, as both may be compounded with Z into a force equal and parallel to Z; the effect of the moment M being to remove its point of application from the origin, to a point of the axis of x, whose distance from the origin is

$$x = \frac{\mathrm{M}}{\mathrm{Z}}.$$

and that of the moment L, to remove it to a point in the axis of y, whose distance from the origin is

$$y = \frac{\mathrm{L}}{\mathrm{Z}}.$$

and the effect of both, conjointly, is to remove the point of application of Z, to the extremity of the diagonal of a rectangle, whose sides, measured from the origin on the axes of x and y, are

$$\frac{\mathrm{M}}{\mathrm{Z}}. \qquad \frac{\mathrm{L}}{\mathrm{Z}}.$$

This point must not be exterior to the area of the triangle made by joining the three points of support: and on this principle, the limitations by which the magnitudes of L and M are restricted, are easily found.

The pressures on the three points are known, when the point of application of the general resultant is known; and this is given by its coordinates, viz.

$$x = \frac{\text{M}}{\text{Z}}. \qquad y = \frac{\text{L}}{\text{Z}}.$$

Let it be the point *o* within the area of the triangle *abc*, (Fig. 33.) formed by joining the points of support. The force applied at *o*, is the force Z, and this is to be distributed on the points *a*, *b*, *c*. Therefore, drawing the line *ao*, meeting the opposite side at *d*, the pressure Z may be distributed on the points *a* and *d*; the parts being, respectively,

$$\text{Z}.\frac{od}{ad}. \qquad \text{Z}.\frac{ao}{ad}.$$

The latter of these is distributed on the points *b* and *c*, in the ratio of *dc* to *db*. Wherefore, if from *o*, lines are drawn to the angles of the triangle, resolving the whole area into three triangular areas; the pressures on the three points shall be proportional to the areas of these triangles; that on each point being represented by the area of the triangle placed on the opposite side. For if the total pressure Z, is represented by the area of the entire triangle *abc*, that on the point *a*, shall be represented by the area *boc*; this being to the whole area, in the ratio of *od* to *ad*: and therefore, the pressure on *d*, shall be represented by the sum of the two remaining triangles *boa*, *coa*. But the pressure on *d* is divided at *b* and *c*, in the ratio of *dc* to *db*; *i. e.* of the triangle *coa*, to the triangle *boa*.

When the body touches the plane in more than three points, the resistances, when excited, must compound a force perpendicular to the plane, at some point within the polygon formed by joining the points of contact; omitting those points at which the angles are re-entering. Wherefore, the forces of the system, to be capable of being equili-

brated with such resistances, must have a single resultant directed perpendicularly to the plane, at some point within the area of that polygon; and when this condition is satisfied, the equilibrium is necessarily established by the resisting forces. For the resultant of the system may be resolved into parallel forces bearing on three or more of these points, where they will be necessarily equilibrated.

In all cases, the sum of the pressures is the sum of the forces of the system resolved in directions perpendicular to the plane; but if the points of contact are more than three, the distribution of the pressures is indeterminate. This is no more than must have been expected: for the point of application of the general resultant being within the area of the polygon, it is within the area of some one of the triangles into which the polygon is resolvable. Accordingly, the pressure may be totally distributed on the angles of this triangle. The pressures thus supported, may then be variously transferred to lines drawn between the other angles of the polygon, and intersecting the sides of that triangle; and the pressures laid on each of these lines, divided on their extreme points.

In general; whatever be the nature of the surface or surfaces in contact with the body, and whatever the number of the points of contact, the equilibrium will be established by means of the resistances, whenever the forces of the system are reducible to another system of forces, bearing perpendicularly against any number of those points, and not otherwise: and the pressure at each point will be the force directed against it, unless so far as this pressure may be distributed, in the whole or in part, on any of the other points of support.

This account of the pressures relates to a system in equilibrio. To estimate the strains on the several points of a

system in motion, the forces required to produce those motions are also to be considered. The actions of the several points are then the resultants of those forces and of the resistances here treated of; and the strains are equal and opposite to these resultants.

SECTION VI.

OF THE CENTRE OF GRAVITY.

1. The theorems already established, relative to forces applied to a point or to a system of points, are altogether independent of the particular sources from whence those forces are derived. Those theorems are, therefore, equally true, whether the forces are those supplied by animal strength, or those by which the particles of matter naturally influence each other. Of this latter kind is the force of gravity, and as this constantly offers itself to our notice, as one of the conditions from which the solution of mechanical questions is to be derived, it becomes requisite, in an especial manner, to consider the laws of its action on terrestrial bodies, and the manner in which it is to be treated.

It is well known of all bodies near the earth's surface, that they tend to descend, each in a direction perpendicular to the horizon of the place; and that when this tendency is counteracted, a pressure is sustained, which is denominated the weight of the body. The descent of bodies when free to move, and the pressure when the motion is restrained, are, both of them, manifest indications of the action of a force; and this force is denominated the force of gravity.

The force of gravity accelerates all bodies equally; *i. e.* it generates in them equal velocities in equal times; for it is found that in vacuo, the lightest feather and the most ponderous substance descend together as if they were parts of the same mass. And the same thing is yet more satisfactorily established by observations made with the pendulum, as shall be shown hereafter. This being admitted, it follows that the force of gravity acts equally on all the units of mass,

and in parallel directions. Thence arises a system of equal and parallel forces applied to the several units of mass of which the body consists; and the resultant of these is equal to their sum. Wherefore, denoting the force of gravity acting on the unit of mass, *i. e.* its weight by g, and the mass of the body, *i. e.* the number of units in that mass, by M, and putting W for the weight of the body, there is

$$W = Mg. \quad (1)$$

Hence, g being the same for all bodies, "The weights of bodies are proportional to their quantities of matter."

From the foregoing equation it appears that the "force of gravity" and "the weight of a body" are not expressions of the same import. By the former is meant the intensity of the power as it exists in nature, whose measure is the force with which it acts on the unit of mass; by the latter is meant the force of gravity as applied to the particular body under consideration; and this depends not only on the intensity of the force as it exists in nature, but also on the number of such units in the mass of the body on which it is exerted.

The quantities of matter in bodies being proportional to their weights, and the weights of bodies of the same bulk being exceedingly various, it is evident that there must exist the widest differences, with respect to the condensation of the matter of which they consist. The relation between the quantity of matter in a body and its bulk or volume, is that which is meant by its density; and, therefore, the densities of bodies are as the quantities of matter contained in a given volume. But in comparing the densities of different substances it is found convenient to refer them all to the same scale, agreeably to what has been done with respect to the quantities which enter into equation (1). This is at once effected by fixing on some one substance, whose density is to be regarded as the standard of comparison, and, therefore, as the unit of the scale of densities. Various considerations

concur, to render water the most eligible for this purpose. But the density of water itself is known to vary, according to the quantity of foreign matter which it may hold in solution; and also, according to its different degrees of temperature. The uncertainty which would follow from the former source is avoided, by taking distilled water; and that from the latter, by taking the water at its maximum of condensation, which corresponds to a temperature of 40° of Fahrenheit, or somewhat more than 4° of the centigrade thermometer. Therefore, putting unity for the density of distilled water of this temperature; the densities of bodies, which in the same volume, contain twice or thrice the quantity of matter, are denoted by the numbers 2, 3, and so forth; the quantities of matter being compared, by means of the weights to which they are proportional.

When, therefore, the quantities of matter are estimated by the volumes, it is on the supposition that the densities are equal; but if these are unequal, the numbers which express the ratio of the volumes, must be multiplied by those which denote the densities: and the quantities of matter in different bodies are proportional to these products. This is expressed by the equation

$$\text{M} = \text{VD} \quad (2)$$

in which M, V, and D are numbers denoting the ratios of the quantities for which they stand to their several units of measure. Thus, if a cubic inch is taken for the unit of volume, and the density of water for the unit of density, then the quantity of matter in a cubic inch of water shall be the unit of mass: and for any body whatsoever, in the last equation, V shall denote the number of cubic inches in its bulk or volume, D its density with respect to that of water, and M its mass, in relation to that of a cubic inch of this fluid. Substituting in equation (1) this value of M, there is

$$\text{W} = \text{V}.\text{D}.g. \quad (3)$$

wherein g denotes the weight of a cubic inch of water.

This weight is converted into grains by putting for g in the last product 252.952, the number of grains contained in that volume of water.

2. As in all positions of the body with respect to a horizontal plane, the same forces of gravity are applied to the same points, and in parallel directions, these forces must have a centre, *i. e.* in a body, considered as an invariable system of points, there is a certain point through which, in all positions of the body, the resultant of those forces passes; and to which point it may therefore be supposed to be immediately applied. This point was generally named the centre of parallel forces; but when the forces are the gravitations of the constituent parts, it is especially designated by the name of the centre of gravity.

This reduction of the weights of the several molecules of a body to a single resultant applied to an invariable point, is of great use in the solution of all questions in mechanics, wherein the weights of bodies are concerned: for instead of an indefinite number of forces and their points of application, we have only to consider a single force, which is the weight of the body, and its point of application, which is the centre of gravity; and having added this force to the others given by the conditions of the question, we can proceed to apply the statical theorems already established relative to forces acting on a system of material points destitute of gravity.

In seeking the centre of gravity of a body, it is to be observed, that if the density is the same throughout the mass, the equal forces are applied at equidistant points, in which case the position of the centre of gravity will depend altother on the figure: and this supposition of homogeneity is always made in the investigation of the centres of gravity of geometrical figures. But if the density is not uniform, the body being supposed to be divided into particles of a given mass, the number of these contained within a given space will be various: *i. e.* the points to which equal forces are applied will be more condensed in one part of the body than in

another; in which case, the position of the centre of gravity shall depend, not on the figure only, but also on the way in which the matter is distributed throughout the volume, *i. e.* on the relative densities of the parts.

This being premised, we shall proceed to illustrate the methods by which the centre of gravity of a body or system of bodies is ascertained, and then to give an account of the chief properties of this point.

3. The methods of finding the centre of gravity are the same as those given for finding the centre of parallel forces, the symbols P, P′, P″, &c. by which the parallel forces had been denoted, now denoting the weights or the masses to which they are proportional.

According to the first of these methods, described in Sect. II. Art. 6. the distance between two elements of the system is divided in the inverse ratio of the masses or weights: the point, by which the connecting line is so divided, is the centre of gravity of those two elements, whose sum is then to be regarded as concentred in this point. The line connecting this point with a third element is, in like manner, to be divided in the inverse ratio of the weights connected by it. The common centre of gravity of three elements being thus found, and the sum of their weights being supposed concentred in it, the same process is to be continued until all the weights are united in one point, which will be the common centre of gravity of the entire system.

If the masses are equal, the rule now delivered will be reduced to the following. The line connecting two of those masses is to be bisected. The line connecting this point of bisection with a third of those masses is to be divided in the ratio of 1 : 2. That connecting this point of division with a fourth of the masses is to be divided in the ratio of 1 : 3. and so-forth, *i. e.* one-half is to be taken from the line connecting the two first points; one-third from the line connecting the point thus found, with a third of the given points; one-fourth from the line connecting the point thus found, with a fourth

of the given points, and so-forth. The points thus successively found are the centres of gravity of the parts so combined; and the last, the common centre of gravity of the whole system.

If the body, whose centre of gravity is sought, is of considerable magnitude, the subdivision into parts, such as might be regarded as physical points, and the treatment of these, according to the method now described, might be exceedingly tedious. But if the centre of gravity of any portion of the body is already known, the labour of computation is diminished, by supposing the weight of that portion concentred in its proper centre of gravity.

Thus, if an indefinite number of material points of equal weight were uniformly arranged along the sides of a polygon, the centre of gravity of those in any one line is the middle point of that line: and by supposing the weights, in each line, united at the middle point of that line, the problem is reduced to that of finding the common centre of gravity of a system of weights, equal in number to the sides of the polygon; the weights themselves being proportional to those sides, and applied at their middle points.

If a physical surface is symmetrically divided by a certain right line, the centre of gravity is in this line: and if it is symmetrically divided by another line also, the centre of gravity is, at once, known to be at the intersection of the lines by which it is so divided. Also, if a solid is symmetrically divided by a certain plane, the centre of gravity is in that plane; and if it is divided in like manner by a second plane, the centre of gravity is in both planes, and therefore, in the line of intersection: and if it is so divided by a third plane, the centre of gravity is at once determined to the single point common to the three planes. Thus the area of the triangle ABC, being divided into elementary trapezia, or physical lines parallel to any one of its sides BC, as in (Fig. 34.); the centre of gravity of each physical line is at its middle point, where accordingly its weight may be supposed to be applied: and

as all such points of bisection are situated in the line drawn from the angle A to D, the middle point of the opposite side BC, it is plain that the common centre of gravity of all these elements, or the centre of gravity of the area of the triangle is in the line AD. For the same reason it is in the line BE, (Fig. 35.) drawn from another of the angles B, to E, the middle point of the opposite side AC: and therefore it is at g, the point where the lines AD, BE, intersect. It remains only to determine the distance of g, from any one of the angles, measured on the right line drawn from that angle to the middle point of the opposite side. Now the sides AC, BC, being bisected at E and D, the line ED is parallel to the third side AB: wherefore, the triangles CED, CAB, as likewise the triangles EgD, AgB, are similar. The latter pair give the following proportion:

$$\mathrm{A}g : g\mathrm{D} :: \mathrm{AB} : \mathrm{ED}.$$

and the former give

$$\mathrm{AB} : \mathrm{ED} :: \mathrm{BC} : \mathrm{DC} :: 2 : 1.$$

wherefore,

$$\mathrm{A}g = 2g\mathrm{D}.\ i.\ e.\ \mathrm{A}g = \tfrac{2}{3}\mathrm{AD}.$$

The centre of gravity of the area of a triangle is then the same as that of three equal bodies, applied at its three angular points; since it is found by the same rule, viz. by bisecting one of the sides, and dividing the line drawn from the point of bisection to the third angle, in the ratio of 1 : 2.

Knowing how to find the centre of gravity of a triangle, that of any right lined polygon is readily found; inasmuch as any such polygon may be resolved into triangles, whose weights, proportional to their areas, may be supposed concentred in their respective centres of gravity. The problem is then reduced to that of finding the common centre of gravity of so many weights applied at these points; which may be done by the method of successive composition already described.

A triangular pyramid being resolvable into physical surfaces parallel to any one of the triangular faces, it is evident that the centres of gravity of these surfaces are arranged in the same right line, drawn to the centre of gravity of the parallel face from the opposite angle. For the like reason, it must be also in the right line drawn to the centre of gravity of any of the other triangular faces, from the angle opposed to it; and, therefore, it must be at the intersection of two such lines.

Thus, in (Fig. 36.) bisecting DC in E, and in the line BE, drawn from the opposite angle of the triangle, taking $Bg = \frac{2}{3}BE$, the point g is the centre of gravity of the triangular face BDC; and that of the pyramid is in the right line Ag, drawn to this point from the opposite angle. In like manner, taking $Ag' = \frac{2}{3}AE$, and drawing Bg'; the centre of gravity of the pyramid is also in this line Bg'. Now as Ag, Bg', are in the same plane, viz. that of the triangle AEB, they must intersect as at g'': wherefore, this point g'', is the centre of gravity of the pyramid.

All that now remains, is to ascertain the distance of the point g'', from either of the angles A or B. In order to this, it is to be observed, that $Eg' : EA :: Eg : EB$; wherefore, the lines $g'g$ and AB are parallel: and therefore the triangles $g'gE$, ABE, as also the triangles $gg'g''$, $Ag''b$, are similar: accordingly,

$$Ag'' : g''g :: AB : g'g :: AE : g'E :: 3 : 1.$$

which gives

$$Ag'' = 3 . g''g = \tfrac{3}{4}Ag.$$

Showing that the centre of gravity of a triangular pyramid is in the right line, drawn from any one of the angles to the centre of gravity of the opposite face; at a distance from the angle which is $\frac{3}{4}$ of this line.

The centre of gravity of a triangular pyramid is then the common centre of gravity of four equal bodies, applied at its

four solid angles; those centres being found by the same rule. In both cases, the line connecting two of the angles is to be bisected: the line drawn from this point of bisection to a third angle, is to be divided in the ratio of one to two: and finally, the line connecting the last point of division with the fourth angle, is to be divided in the ratio of one to three.

The rule for determining the centre of gravity of a triangular pyramid, may be extended to pyramids standing on polygonal bases. For any such polygon may be resolved into triangles; and, therefore, the pyramid into triangular pyramids, having the same summit, and those triangles for their bases. The centre of gravity of each of the triangular pyramids, and, therefore, that of the entire pyramid, shall be in a plane parallel to the base, at a distance from the summit equal to $\frac{3}{4}$ of the altitude. It shall also be in a right line drawn from the summit to the centre of gravity of the base. Wherefore, it is at the point of this line where it is intersected by that plane, *i. e.* at a distance from the angle equal to $\frac{3}{4}$ of its length.

In prisms, this resolution into pyramids is unnecessary; for the sections parallel to the bases being all similar, their centres of gravity are in the same right line; and these sections being moreover equal, their common centre of gravity is evidently the middle point of the line, which joins the centres of gravity of the parallel bases.

What has been proved relative to the centre of gravity of a pyramid, standing on a polygonal base, is evidently applicable to a cone; which may be regarded as the limit of the inscribed or circumscribed pyramids, and in which limit they end, when the sides become infinite in number. Wherefore, the centre of gravity of a cone is in the axis, at a distance from the summit equal to $\frac{3}{4}$ of its length.

In like manner, what has been shown of prisms may be extended to cylinders standing on any curvilinear bases whatever.

The centre of gravity of a sphere or spheroid is obviously at the centre; as each of these figures may be divided symmetrically by three planes, all passing through this point.

4. The method of finding the centre of gravity, by reference to three coordinate planes, is that explained in Sect. II. Art. 7.

If the body, or system of bodies, whose centre of gravity is sought, is divided into equal masses, we shall have

$$\mathrm{P} = \mathrm{P}' = \mathrm{P}'', \&c.$$

wherefore, putting n for the number of these equal masses, those theorems will become

$$\mathrm{X} = \frac{x + x' + x'' + x''' + \&c.}{n}.$$

$$\mathrm{Y} = \frac{y + y' + y'' + y''' + \&c.}{n}.$$

$$\mathrm{Z} = \frac{z + z' + z'' + z''' + \&c.}{n}.$$

The equations may be always presented in this form, and they show that the distance of the centre of gravity from any plane, is the sum of the distances of the equal masses divided by their number: or that it is the mean distance of the weights of the several parts, and, therefore, that the centre of gravity is the centre of the mass.

If the body is homogeneous, *i. e.* of uniform density, the equal masses, into which it is divided, will have equal volumes; and the centre of gravity will be then the centre of magnitude. From this and Art. 3. it appears, that the distance of the centre of gravity of a triangle from any plane, is the mean distance of its three angles from the same. And that the distance of the centre of gravity of a triangular pyramid from any plane, is the mean distance of its four angles.

The formula of Sect. II. Art. 7. may be presented in a more abridged form, by putting

$$\Sigma(\text{P}.x), \quad \text{for } \text{P}.x + \text{P}.'x' + \text{P}.''x'' + \&\text{c}.$$

the character Σ, denoting the sum of all quantities corresponding to that to which it is prefixed. In the same manner, $\Sigma(\text{P})$ is put for $\text{P} + \text{P}' + \text{P}'' + \&\text{c}.$, and the same notation being extended to the other two formulæ, they are thus presented:

$$\text{X} = \frac{\Sigma(\text{P}.x)}{\Sigma(\text{P})}. \qquad \text{Y} = \frac{\Sigma(\text{P}.y)}{\Sigma(\text{P})}. \qquad \text{Z} = \frac{\Sigma(\text{P}.z)}{\Sigma(\text{P})}.$$

It is seldom that we have occasion for all these formulæ: for if the centre of gravity is known to be in a certain plane, its distance from the plane being cypher, the product of this distance into the whole mass is also cypher: and therefore, also, the sum of the products of the partial masses into their respective distances from the same plane is cypher, the sum of the products of those which lie at one side into their distances from the plane, being equal to the sum of the products, similarly made out, for those which lie at the other side of the plane, and with contrary signs. And conversely; if the sum of the products be cypher, the distance of the centre of gravity is also cypher.

When any such plane is known, it may be taken for one of the coordinate planes, as for example, for the plane of xy, and we shall then have

$$\Sigma(\text{P}.z) = 0. \quad \text{Z} = 0.$$

and the problem for finding the centre of gravity is solved by the two first equations, which give its distance from the axes of x and y. Wherefore, "when the centres of gravity of the bodies of a system lie all in the same plane, the common centre of gravity is found by its distance from each of two axes in the same plane; the distance from each axis being had by multiplying each body into the perpendicular distance of its centre of gravity from that axis, and dividing the sum of those products by the sum of the masses."

If the centres of gravity of the bodies, or partial masses, lie in the same right line, by taking this line for the intersection of two of the coordinate planes, as of the planes zx, yx; those centres being in each of these planes, the common centre of gravity shall be in both. Wherefore,

$$\text{Z} = 0. \quad \text{and} \quad \text{Y} = 0.$$

and the third equation will suffice, which gives the distance of the common centre of gravity from a plane perpendicular to the axis of x; *i. e.* from a certain point on that line itself. Wherefore, "when the centres of gravity of the bodies of the system, or of the partial masses, lie all in the same right line; the common centre of gravity is on the same line, and its distance from an assumed point of that line is found, by taking the sum of the products of the several bodies or masses into their respective distances from that point, and dividing by the entire mass.

Let it be proposed to find the centre of gravity of the perimeter of a plane polygon BCDEF, (Fig. 37.) by means of the coordinates of its angles.

The centres of gravity of the several sides, being at the middle points of those sides, are all in the same plane: wherefore, drawing two axes in this plane, viz. Ay, Ax, and denoting by x, y, the coordinates of the point B; by x', y', those of the point C; by x'', y'', those of the point D, &c. and bisecting the sides BC, CD, DE, &c. at the points o, o', o'', &c. the coordinates of the point o, shall be

$$\frac{x+x'}{2}. \qquad \frac{y+y'}{2}.$$

those of the point o' shall be

$$\frac{x'+x''}{2}. \qquad \frac{y'+y''}{2}.$$

those of the point o'' shall be

$$\frac{x''+x'''}{2}. \qquad \frac{y''+y'''}{2}.$$

and so of the rest. Then, putting P, P′, P″, &c. taken proportional to the sides, for the weights applied at the middle points of those sides, and making these substitutions in the general formulæ, we have

$$\mathrm{X} = \frac{\mathrm{P}(x+x') + \mathrm{P}'(x'+x'') + \mathrm{P}''(x''+x''') + \&c.}{2(\mathrm{P}+\mathrm{P}'+\mathrm{P}'') + \&c.}$$

$$\mathrm{Y} = \frac{\mathrm{P}(y+y') + \mathrm{P}'(y'+y'') + \mathrm{P}''(y''+y''') + \&c.}{2(\mathrm{P}+\mathrm{P}'+\mathrm{P}'') + \&c.}$$

Showing, that each of the coordinates of the centre of gravity of the perimeter of a polygon is had, by multiplying each side into the sum of the corresponding ordinates of its extreme points, and dividing the sum of these products by twice the perimeter.

To find the centres of gravity of curvilinear figures, the aid of the integral calculus is generally requisite. The differential of the figure is to be multiplied into each of its three coordinates; each of these products being made a function of a single variable, by means of the equations of the figure, is to be integrated within the proposed limits: the definite integrals thus obtained, are the numerators of the values of X, Y, Z, the coordinates of the centre of gravity. Their common denominator is the magnitude of the figure obtained by integrating its differential in like manner.

Thus, for a curve whose length is s, the general formulæ, by which the coordinates of the centre of gravity are expressed, will be

$$\mathrm{X} = \frac{\int x\,.\,ds}{s}. \qquad \mathrm{Y} = \frac{\int y\,.\,ds}{s}. \qquad \mathrm{Z} = \frac{\int z\,.\,ds}{s}.$$

The differential ds being

$$\sqrt{(dx^2 + dy^2 + dz^2)}.$$

the product of this by its distance from the plane of xy, *i. e.* zds is

$$z.\ \sqrt{(dx^2 + dy^2 + dz^2)}.$$

This being converted into a pure function of z, by means of the equations of the curve, and integrated, the extreme values of z are to be substituted; the difference of the resulting quantities is the numerator in the value of Z. Its denominator, s, is the definite integral of

$$\sqrt{(dx^2 + dy^2 + dz^2)}.$$

to be obtained in the same way. The values of X and Y, are found by a similar process.

If the curve is of single curvature, it is evident from what has been already observed, that the problem may be solved by two of the equations: and if it is situated symmetrically with respect to a certain right line, that one of those equations will suffice.

For example; let it be proposed to find the centre of gravity of a circular arc.

The arc being placed symmetrically with respect to the radius drawn to its middle point, this radius may be taken for the axis of x; and the problem will be solved by means of the equation

$$X = \frac{\int x . ds}{s}.$$

in which,

$$ds = \sqrt{(dx^2 + dy^2)}.$$

The origin being taken at the centre, the equation of the curve is

$$y^2 = r^2 - x^2.$$

which gives,

$$dx^2 = \frac{y^2}{x^2} . dy^2.$$

Substituting this value, and integrating, there is

$$\int x . ds = r . y + C.$$

Let b and $-b$, be the values of y, at the extreme points

of the arc; these are to be successively substituted; and one of the resulting quantities being subtracted from the other, the definite integral is $2r.b$. Wherefore,

$$\text{X} = \frac{2b.r}{s}.$$

showing, that the distance of the centre of gravity of the arc from the centre, is a fourth proportional to the arc itself, its chord and radius.

For the semicircle,

$$s = \pi r. \qquad b = r. \qquad \text{giving X} = \frac{2r}{\pi}.$$

and therefore,

$$r = \frac{\pi}{2}. \text{ X}.$$

which shows, that the radius of a circle is equal to the quadrantal arc of another circle, whose radius is the distance of the centre of gravity of half the periphery of the former from its centre.

Let it be proposed to find the centre of gravity of a cycloidal arc.

This curve is generated by the motion of a point taken in the periphery of a circle, whilst the circle revolves in its own plane, on a right line.

Let AB (Fig. 38.) be the line on which the circle revolves, or the base of the cycloid; V its vertex or middle point; VD its axis, which is also the diameter of the generating circle. It is evident that the two branches, VA, VB, are symmetrical about the axis; and that if equal arches are measured from V, their centres of gravity shall be equidistant from the axis VD, and also from the base. Wherefore, the common centre of gravity of the two equal arcs shall be at the same distance from the base, and in the axis VD. Accordingly, taking V for the origin, and VD for the axis of x, it will be sufficient to find the centre of gravity of an arch at either side of VD.

For any point, m, there will be

$$\text{A}p = \text{arc}\, pm = \text{arc}\, \text{D}r.$$

therefore,

$$p\text{D} \text{ or } mr = \text{arc V}r.$$

and

$$mq = \text{arc} = \text{V}r + rq.$$

i. e. putting a, for VD,

$$y = \text{arc (verse sine} = x) + \sqrt{(ax - x^2)}.$$

and

$$dy = \frac{dx}{\text{sin.arc V}r} + \frac{\frac{a}{2} - x}{\sqrt{(ax - x^2)}}.\, dx$$

$$= \frac{\frac{a}{2}}{\sqrt{(ax - x^2)}}.\, dx + \frac{\frac{a}{2} - x}{\sqrt{(ax - x^2)}}.\, dx.$$

that is,

$$dy = \frac{(a - x)\, dx}{\sqrt{(ax - x^2)}} = \frac{\sqrt{(a - x)}}{\sqrt{x}}.\, dx.$$

Accordingly,

$$ds = \sqrt{(dx^2 + dy^2)} = \sqrt{\frac{a}{x}}.\, dx. \quad \text{and } s = 2\sqrt{ax}.$$

Also,

$$\int x ds = a^{\frac{1}{2}} \int x^{\frac{1}{2}}.\, dx = \frac{2}{3} a^{\frac{1}{2}}.\, x^{\frac{3}{2}}.$$

In these integrations, no constant quantities are added, as both members of each equation vanish together, for $x = 0$.

$$\text{X} = \frac{\int x . ds}{s} = \frac{\frac{2}{3} a^{\frac{1}{2}}.\, x^{\frac{3}{2}}}{2 a^{\frac{1}{2}}.\, x^{\frac{1}{2}}} = \frac{x}{3}.$$

which shows, that if a line is drawn through the vertex, pa-

rallel to the base, the perpendicular on that line, from the centre of gravity of an arch measured from the vertex, is one-third of the perpendicular distance of its extreme point from that same line; and, therefore, that the centre of gravity of a cycloidal arc bisected at the vertex, is in the axis, at a distance from the vertex equal to one-third of the sagitta of the arc: and that the distance of the centre of gravity of the entire curve from the same point, is $\frac{a}{3}$.

The differential of a plane surface is ydx; wherefore, the formulæ for its centre of gravity are,

$$X = \frac{\int x.y.dx}{\int y.dx}. \qquad Y = \frac{\int y^2.dx}{\int y.dx}.$$

If the surface is symmetrical about a certain right line, it is evident that it may be resolved into elements, whose centres of gravity are in that line: and that if the same line is taken for the axis of the abscissæ, the first of these equations will suffice.

Example.—Let it be proposed to find the centre of gravity of a trapezium ABCD, (Fig. 39.)

If the trapezium is resolved into elements by lines parallel to AD, it is evident that the right line EF, which bisects the parallel sides AD, BC, shall pass through the centre of gravity of each element: wherefore, it will be convenient to make it the axis of the abscissæ. Let F be the origin, $EF = a$; $AD = b$; $BC = b'$; $Fm = x$; $mn = y$. Then,

$$2y = \frac{a.b' + (b - b')x}{a}.$$

If the angles at A and D are equal, y shall be perpendicular to the axis of x; and the element of the area shall be

$$2y.dx = \frac{[a.b'. + (b - b')x]\,dx}{a}.$$

and

$$2\,x.y.dx = \frac{[a.b'.x + (b-b')\,x^2]\,dx}{a}.$$

Wherefore,

$$2\int y.dx = \frac{2\,a.b'.x + (b-b')\,x^2}{2\,a}.$$

and

$$2\int x.y.dx = \frac{3\,a.b'.x^2 + 2\,(b-b')\,x^3}{6\,a}.$$

and

$$\text{X} = \frac{2\int x.y.dx}{2\int y.dx} = \frac{3\,a.b'.x + 2\,(b-b')\,x^2}{6\,a.b' + 3\,(b-b')\,x}.$$

In these integrations, no constant quantity was added; because, for $x = 0$. both members of each equation vanish together.

If the angles at A and D are unequal; or which is the same thing, if the line EF is oblique to the parallel sides, AD, BC, the same determination will serve; as will appear by putting β for the angle FED, contained between the axis and the ordinates. Then, each element is

$$2\sin.\beta.y.dx.$$

and

$$\text{X} = \frac{2\sin.\beta.\int x.y.dx}{2\sin.\beta.\int y.dx} = \frac{\int x.y.dx}{\int y.dx}.$$

Taking the value of X for $x = a$. it is

$$\text{X} = \frac{a\,(b' + 2\,b)}{3\,(b' + b)}.$$

This solution may serve for a triangle or parallelogram. For the former, by making $b' = 0$. in which case,

$$\text{X} = \frac{2}{3}\,a.$$

and for the latter, by making $b' = b$; in which case,

$$\mathrm{x} = \frac{a}{2}.$$

Example.—Let it be proposed to find the centre of gravity of the segment of a circle, whose equation, taking the centre for the origin, is

$$y^2 = r^2 - x^2.$$

The centre of gravity of the segment is evidently in the radius by which the arc is bisected. Wherefore, taking this for the axis of the abscissæ,

$$\mathrm{x} = \frac{2\int x.y.dx}{\text{area}} = \frac{2\int \sqrt{(r^2 - x^2)}\,x.\,dx}{\text{area}}. =$$

$$-\frac{\frac{2}{3}(r^2 - x^2)^{\frac{3}{2}} + \mathrm{c}}{\text{area}}.$$

Taking the integral from $x = a$ to $x = r$; in which last case its value is c, the value of y being then cypher, we have

$$\mathrm{x} = \frac{\frac{2}{3}(r^2 - a^2)^{\frac{3}{2}}}{\text{area}}.$$

For the centre of gravity of the area of the semicircle, making $a = 0$; and putting for the area, its value, $\frac{\pi r^2}{2}$; there is for this case,

$$\mathrm{x} = \frac{4\,r}{3\pi}.$$

The sector cmvn, (Fig. 40.) is composed of the segment mvn, and of the triangle mcn. The centre of gravity of the triangle is in the same radius, by which the arc mn is bisected; and at a distance from c equal to $\frac{2}{3}$ cd. Wherefore, the distance between the centres of gravity of the triangle and circular segment is known; and if this distance is divided in the inverse ratio of the areas, the point of division will be

the centre of gravity of the sector. This process is, however, unnecessary. For if with the centre C, and a radius equal to $\frac{2}{3}$ Cr, an arc is described, bounded by the lines Cm, Cn, this arc is the locus of the centres of gravity of the indefinitely small and equal sectors into which the sector Cmvn is resolvable; those elementary sectors being regarded as right lined triangles. Moreover, these elementary sectors being equal, and their centres of gravity being uniformly arranged along this arc, their common centre of gravity, *i. e.* the centre of gravity of the entire sector, must be that of the arc: hence, it follows, that the distance of the centre of gravity of a sector, from the centre of the circle, is a fourth proportional to the arc, its chord, and $\frac{2}{3}$ of its radius.

Example.—For the centre of gravity of the segment of a parabola, whose equation is $y^2 = px$, the diameter being taken for the axis of the abscissæ, and the vertex for the origin, we have

$$\frac{\int y.x.dx}{\int y.dx} = \frac{3}{5}x.$$

In these integrations, no constant quantity is added, because for $x = 0$. each integral is, evidently, cypher.

It is also evident from the observation already made, that the same formula will serve whether the ordinates are perpendicular or oblique to the diameter, *i. e.* whether the diameter which bisects the chord of the segment is the axis or not.

In figures of revolution, the axis of rotation being taken for the axis of the abscissæ, it is plain that the first of the three general equations of Art. 4, will serve for the determination of the centre of gravity.

If it is proposed to find the centre of gravity of a surface of revolution, the element of the generating curve is

$$ds = \sqrt{(dx^2 + dy^2)};$$

and the element of the generated surface is

$$2\pi y \sqrt{(dx^2 + dy)}.$$

Wherefore,

$$\text{x} = \frac{\int xy \sqrt{(dx^2 + dy^2)}}{\int y \sqrt{(dx^2 + dy^2)}}.$$

in which, substituting for y, dy, their values in x, collected from the equation of the generating curve, and integrating within the proposed limits, the value of x is obtained.

Example.—The surface of a truncated cone is generated by the revolution of a right line. Let CD be that line, (Fig. 41.) and AB the axis round which it revolves. Then, putting a for AB; b for BD; and b' for AC, there will be

$$y = \frac{a.b' + (b - b')x}{a}.$$

and

$$dy = \frac{(b - b')\,dx}{a}.$$

Therefore,

$$\sqrt{(dx^2 + dy^2)} = \frac{\sqrt{[a^2 + (b - b')^2]}\,dx}{a}.$$

The invariable coefficient of dx, occurring in the numerator and denominator of the value of x, may be suppressed. Wherefore,

$$\text{x} = \frac{\int y.x.\,dx}{\int y.dx}.$$

But the section of the conic frustum, in which is the axis, is a trapezium, whose centre of gravity is in the axis of the cone, at a distance from the lesser base expressed as above; whence it follows, that the centre of gravity of the surface of the conic frustrum is the same as that of the section of the solid, in which is the axis: and also, that the centres of gravity of the surfaces of a cone and cylinder are the same as those of the sections in which are their axes.

Example.—Let it be proposed to find the centre of gravity of a spheric zone, generated by the revolution of the circular arc *mi*, about the radius c*v*. (Fig. 40.)

Making this radius the axis of the abscissæ, and the centre c the origin, the equation of the circle is

$$y=\sqrt{(r^2-x^2)}.$$

Wherefore,

$$\sqrt{(dx^2+dy^2)}=\frac{rdx}{\sqrt{(r^2-x^2)}}.$$

and

$$\int y.\sqrt{(dx^2+dy^2)}=rx+\text{c}.$$

$$\int xy\ \sqrt{(dx^2+dy^2)}=\frac{rx^2}{2}+\text{c}'.$$

Then, putting a for c*d*; a' for c*e*, and taking the integrals between these limits, there will be

$$\text{x}=\frac{a'^2-a^2}{2\,(a'-a)}=\frac{a'+a}{2}.$$

Which shows, that the centre of gravity of the zone is the middle point of the portion of the axis intercepted between its two circular bases.

For the spheric segment there is $a'=r$; and in that case,

$$\text{x}=\frac{r+a}{2}.$$

showing, that the centre of gravity of the surface of a spheric segment is at the middle point of the sagitta.

In general, the equation of the surface of any geometric figure being of this form

$$dz=p.dx+q.dy.$$

in which p and q are the partial differences of z, with respect to x and y, the differential of the surface is

$$\sqrt{(1+p^2+q^2)}\,dx.dy.$$

wherefore,

$$\text{X} = \frac{\int\int x\sqrt{(1+p^2+q^2)}\,dx.dy}{\int\int\sqrt{(1+p^2+q^2)}\,dx.dy}.$$

$$\text{Y} = \frac{\int\int y\sqrt{(1+p^2+q^2)}\,dx.dy}{\int\int\sqrt{(1+p^2+q^2)}\,dx.dy}.$$

$$\text{Z} = \frac{\int\int z\sqrt{(1+p^2+q^2)}\,dx.dy}{\int\int\sqrt{(1+p^2+q^2)}\,dx.dy}.$$

The double integrals are to be taken within the proposed limits.

A solid of revolution is resolved, by planes perpendicular to the axis, into cylinders or conic frusta, whose altitude is dx. The value of one of these elements is

$$\pi y^2 dx.$$

and the product of this into the abscissa is

$$\pi x y^2 . dx.$$

Wherefore,

$$\text{X} = \frac{\int y^2 x dx}{\int y^2 dx}.$$

in which, substituting for y its value in x, and integrating as before within the proposed limits, the definite value of X is obtained.

Example.—Let it be proposed to find the centre of gravity of an oblong ellipsoid, terminated by two planes perpendicular to the greater axis.

Putting a for the greater, β for the lesser semiaxis; a and a', as before, denoting the distances of the planes from the centre, the equation of the generating ellipse is

$$y^2 = \beta^2 - \frac{\beta^2}{a^2}x^2.$$

wherefore,

$$\int y^2 dx = \frac{\beta^2}{3a^2}(3a^2 x - x^3) + \text{C}$$

$$\int y^2 x dx = \frac{\beta^2}{4a^2}(2a^2x^2 - x^4) + c'$$

These integrals taken between the limits a' and a, give

$$\text{x} = \frac{6a^2(a'^2 - a^2) - 3(a'^4 - a^4)}{12a^2(a' - a) - 4(a'^3 - a^3)}.$$

This expression, being totally independent of the lesser axis of the ellipse, is applicable to the segment of a sphere.

If the ellipsoid were described about the lesser axis, the expression for the distance of the centre of gravity would differ from this only in the change of a for β. Whence it follows, that if the axis of the ellipsoid, whether oblong or oblate, is made the diameter of a sphere, the segments of these figures included between two planes perpendicular to the common axis, shall have their centres of gravity at the same point.

If the section, perpendicular to the axis of revolution, passes through the centre, the distance of the centre of gravity of the semiellipsoid, or hemisphere, will be had from the foregoing, by making $a = 0$. And $a' = a$ for the oblong, or $a' = \beta$ for the oblate ellipsoid, which gives

$$\text{x} = \frac{3}{8}a$$

for the one, and

$$\text{x} = \frac{3}{8}\beta$$

for the other.

Example.—Let it be proposed to find the centre of gravity of a paraboloid, generated by the revolution of a parabolic segment, whose equation is

$$y^2 = px.$$

substituting this value of y^2 in the formula, there is

$$\text{x} = \frac{\int x^2 dx}{\int x dx} = \frac{2}{3}x.$$

In this equation there is no constant quantity.

If the segment, whose centre of gravity is sought, is included between two parallel planes, at the distances a and a', measured from the vertex; these values of x, being successively substituted in each of the integrals, the difference of the results must be taken for the numerator and the denominator of the value sought; thus,

$$\mathrm{X} = \frac{2}{3}\cdot\frac{a'^3 - a^3}{a'^2 - a^2}.$$

These examples may suffice to show how the integral calculus is applied, to solve the problem for solids of revolution, which figures are, evidently, capable of being resolved into elements, whose centres of gravity are in a right line. The centre of gravity of the entire solid being, then, on the same line, its place is determined according to a single formula.

The general method to be used when such facilities do not occur, is to resolve the solid into elements, by planes parallel to each of the three coordinate planes. The expression for an element will then be

$$dx.dy.dz.$$

and that of the volume,

$$\int\int\int dx.dy.dz.$$

and the three coordinates of the centre of gravity of the solid are found, according to the three following formulæ:

$$\mathrm{X} = \frac{\int\int\int .x.dx.dy.dz}{\int\int\int .dx.dy.dz}.$$

$$\mathrm{Y} = \frac{\int\int\int .y.dx.dy.dz}{\int\int\int .dx.dy.dz}.$$

$$\mathrm{Z} = \frac{\int\int\int .z.dx.dy.dz}{\int\int\int .dx.dy.dz}.$$

The integral in each numerator and denominator is to be taken within the proposed limits.

The centre of gravity of any one body of a given system

is immediately found, if the centres of gravity of the remaining bodies, as well as the common centre of gravity of the system are known. For, having

$$mx + m'x' + m''x'', \&c. = \text{MX}.$$

the ordinate of the centre of gravity of any one of the bodies, as of m, is given by the equation

$$x = \frac{\text{MX} - (m'x' + m''x'', \&c.)}{m}.$$

and the other two coordinates of this centre are found in the same manner.

When the centre of gravity of a body, and that of any part of the same are known, that of the remaining part is also known. For in this case, the centres of gravity of the whole and of its two parts are in the same right line; and any two of the masses multiplied into the distances of their centres of gravity from that of the third, give equal products. Thus, if m and m' denote the partial masses, and d, d' the distances of their centres of gravity from the centre of gravity of the sum, which is their common centre of gravity, then $d' = \frac{md}{m'}$. for the distance of the centre of gravity of the mass m', from the common centre of gravity.

Generally, whatever be the point to which, as to an origin, these centres are referred, the coordinates of one of those centres are found from those of the two others, by the following equations:

$$mx + (\text{M} - m)\,x' = \text{MX}.$$

$$my + (\text{M} - m)\,y' = \text{MY}.$$

$$mz + (\text{M} - m)\,z' = \text{MZ}.$$

5. The centre of gravity was shown to a centre of parallel forces, particularized by this circumstance, that the body being resolved into equal masses, the forces applied to those masses are not only parallel, but equal. And as such a

system of forces may be compounded into one parallel force, equal to their sum, which in all positions of the body passes through the same point, denominated the centre of gravity; so, conversely, any single force directed through this point, is resolvable into a system of equal and parallel forces, distributed upon the equal masses, into which the body may be supposed to be divided.

Hence it follows, that there is no tendency to rotatory motion produced in a rigid body, by a force directed through its centre of gravity.

The properties of this centre, usually noticed, relate to a body containing a fixed point, or incumbent on a plane surface. The fixed point is denominated the point of suspension.

The weight of a body, considered as a single force, being applied at the centre of gravity, and directed in the perpendicular towards the horizon; and the resistance or reaction of the fixed point being a force acting in the line by which this point is connected with the centre of gravity, it follows that these two forces cannot be directly opposed, and therefore cannot equilibrate, unless these two lines coincide, *i. e.* unless the fixed point is in the vertical, passing through the centre of gravity; and that then, as the weight of the body is directed through the fixed point, whose reaction is always equal and opposite to the action upon it, the equilibrium is necessarily established.

Accordingly, when the body is suspended by the centre of gravity, the equilibrium is necessarily established, whatever be the position of the body round this point.

But if the point of suspension is different from the centre of gravity, there are but two positions of the body in which the forces are directly opposed to each other, viz. when the centre of gravity is vertically above, or vertically beneath the point of suspension: and these are the two positions of equilibrium.

But equilibrium is of three kinds, viz. stable, unstable, and neutral. It is of the first kind, when the body on any slight change of position, returns to its former position: of the second kind, when the deviation increases: and it is of the third kind, when there is no tendency in the body to recede further from its original position, or to recover that position. It is evident that this last kind can exist, only when the equilibrium continues under a continued change of position; and, therefore, that it can have no place in the case now under consideration, wherein these are but two positions of equilibrium.

To distinguish between the states of equilibrium when the centre of gravity is vertically above or below the point of suspension, it is only requisite to consider the direction of the resultant of the two forces, when the centre of gravity is out of the vertical passing through the point of suspension. Let B be the body; *a* the point of suspension; *o* the centre of gravity; and *ov* the vertical passing through this point. (Fig. 42.) The weight of the body applied at the point *o*, acts in the direction of the line *ov*, and the resistance of the fixed point in the direction of *ao*; towards the fixed point, when the weight acts as a pull, *i. e.* when the angle *voa* is obtuse, as in the figure; and in the opposite direction, when it acts as a pressure, *i. e.* when that angle is acute. The resultant of these two forces must be directed within the angle made by those lines; and, therefore, must tend to carry the centre of gravity downwards to the vertical passing through the point of suspension.

Wherefore, when the centre of gravity is directly above the point of suspension, the equilibrium is unstable: for on the smallest departure from this line it will further descend; and will not rest, until it shall have attained the position vertically beneath the point of suspension. In this latter position, the equilibrium is stable; because on any departure from this position, the centre of gravity will spontaneously return to it.

From these properties of the centre of gravity, this point may be found mechanically.

The property first noticed was, that the body, if suspended by the centre of gravity, will rest in any position indifferently. The point possessed of this property may be determined by trial.

The other property was, that if the body is suspended from a point different from the centre of gravity, the body shall not rest, until that centre attains to the vertical passing through the point of suspension. Accordingly, if the body is freely suspended from two points successively, and the vertical passing through each point of suspension is traced on the body, the intersection of these two lines shall mark the place of the centre of gravity.

When the centre of gravity is in the vertical passing through the point of suspension, the pressure or strain on this point is plainly the weight of the body. And for any position, the strain can be readily determined: for resolving the weight of the body applied to the centre of gravity into two forces: one acting in the line joining that centre with the point of suspension, and the other perpendicular to the same; the former is $\text{w}.\cos.a$, and the latter $\text{w}.\sin a$; a being the angle made by the line drawn to the point of suspension and the vertical. The former of these is destroyed by the resistance of the fixed point, and produces the pressure thereon; the latter is that part of the weight employed in producing motion.

If the body is suspended from two points, it will not rest until the centre of gravity is brought to the vertical plane which contains the two points of suspension; and then the pressures on the two fixed points are given by Sect V. Art. 5. Wherein ε being the angle between the vertical and the line which joins the fixed points,

$$\text{z} = w.\sin.\varepsilon. \qquad \text{x} = w.\cos.\varepsilon.$$

and Y being cypher, the expressions there given are reduced to

$$\frac{w\,[\sin.\,\varepsilon(a-x)+\cos.\,\varepsilon.\,z]}{a}. \quad \text{and} \quad \frac{w\,(\sin.\,\varepsilon.\,x-\cos.\,\varepsilon.\,z)}{a}.$$

These are the pressures in lines perpendicular to that which joins the fixed points. But with respect to the pressures in the direction of that line, all that can be determined about them, on the supposition of the perfect immobility of the points of suspension, is, that the sum of those pressures is $w.\cos.\,\varepsilon$.

If this line is horizontal, the angle ε is right, and therefore,

$$\sin.\,\varepsilon = 1. \qquad \cos.\,\varepsilon = 0.$$

and the pressures altogether vertical; which reduces the expressions for those pressures to

$$\frac{w\,(a-x)}{a}. \quad \frac{w.x}{a}.$$

agreeably to what has been shewn, Sect. II. Art. 4. relative to the resolution of a force into two parallel forces.

If the line which joins the points of suspension is vertical, there will be

$$\sin.\,\varepsilon = 0. \qquad \cos.\,\varepsilon = 1. \qquad \text{X} = w. \qquad \text{Z} = 0.$$

whereby the expressions for the pressures in the perpendicular to the line connecting the fixed points, are

$$\frac{w.z}{a}. \quad \frac{-w.z}{a}.$$

and all that can be known relative to the vertical pressures, is, that their sum is w, or the weight of the body.

This is the case of a door hung on two hinges, wherein a denotes the distance between the hinges; and z the perpendicular on that line from the centre of gravity. So that the upper hinge is drawn out, and the lower hinge pressed

inwards, by a force $w. \frac{z}{a}$. the sum of the vertical pressures being *w*.

For the weight acts in the vertical passing through the centre of gravity, and may therefore be supposed to be applied to any point of this line. This being resolved in the directions of the lines drawn to the hinges from the point so assumed, each of the forces into which it is thus resolved, is again to be resolved in directions vertical and horizontal. The vertical forces will reproduce the weight, and the horizontal forces the strains in their directions. Thus the points of suspension being *m* and *n*; and *o* the centre of gravity, (Fig. 43.) let the lines *no, mo,* be drawn, and let the latter be produced beyond the point *o,* as to *c*. Also, let *ov* be a vertical line directed downwards from the point *o*. The weight which acts in the direction *ov*, is resolved into two forces directed in the lines *oc, on*; the former drawing the upper hinge outwards; and the latter pressing the lower hinge inwards. These three forces are as the sides of the triangle *mon*; *i. e.* the weight being represented by the line *mn,* parallel to its direction, the hinge at *m* shall be drawn out by a force, regresented in magnitude and direction by the line *mo*; and the hinge at *n*, pressed inwards, by a force represented in magnitude and direction by the line *on*. But if each of these strains is to be resolved into two, one in the vertical line *mn,* and the other in the perpendicular to this line; this is done by drawing the perpendicular *od*. The strain *mo* is resolved into *md,* acting vertically downwards; and the strain *do* acting horizontally outwards: also, the strain *on* into *dn,* acting vertically downwards; and *od* acting horizontally inwards. The sum of the vertical pressures is *mn,* or the weight of the door; and this may be sustained by either of the points of suspension, or divided between them in any ratio; but the point *m* is drawn horizontally outwards,

with the force *do*; and the point *n* is pressed horizontally inwards, with an equal force.

The same things would readily follow from the principles established in Sect. III. Art. 3. For the weight being transferred to the vertical containing the hinges, there is generated a pair of equal and contrary forces, whose moment is $w \times od$, which may be turned round in the same plane, so as to become perpendicular to the line *mn*. But the moment $w \times od$ is equal to $\frac{od}{mn} . w \times mn$. *i. e.* to the moment of the forces $\frac{od}{mn} . w$ acting at the interval *mn*.

The body being placed on a horizontal plane, if the vertical passing through the centre of gravity meets the plane in a point within the base, the body shall rest, because the weight applied to the plane at this point, may be distributed on the angles of a triangle within which it is contained: and, by the supposition, such a triangle can be made by joining certain points of support. The weight being so distributed, the pressures are necessarily equilibrated by the resistances of the points to which they are applied. But the stability of the body will depend on the excess of the shortest line that can be drawn from the centre of gravity to the contour of the base, above the perpendicular distance of the same point from the plane. For in order to carry it over the edge of the base, the centre of gravity must be raised to an elevation equal to this difference. Hence in general, the stability of a body is greater, as the distance of the centre of gravity from the plane of support is less, in relation to the extent of the base.

When the base of the body is reduced to a point vertically beneath the centre of gravity, the smallest force, applied in a different direction, should produce a disturbance. Yet even in this case, the equilibrium may be stable, un-

stable, or neutral. Thus, when an elliptic cylinder is placed on a horizontal plane, with its lesser axis in the vertical, the position is that of stable equilibrium. If the same body rests on the extremity of its greater axis, the equilibrium is unstable. A cylinder with a circular base, placed with its side on a horizontal plane, affords an example of neutral equilibrium. The sum of the pressures made at the points of contact, is the weight of the body.

If the body rests on two points of support, the vertical passing through the centre of gravity must meet the line connecting the two points of support at some intermediate point; and the pressure on each point is to the weight, as the distance of the point of intersection from the alternate point, to the distance between the two points of support.

If the body rests on three points, the vertical passing through the centre of gravity must meet the plane within the triangle, formed by joining the three points of support; and then, if lines are drawn to the three angles, from the point vertically beneath the centre of gravity, the pressure on each angular point is to the weight, as the area of the partial triangle placed on the opposite side, to the area of the whole triangle. Sect. V. Art. 7.

If the body touches the plane on which it rests in more than three points, the pressures on the several points are indeterminate, on the supposition of the perfect rigidity of the body, and of the plane on which it rests; and all that can be then determined, with respect to those pressures, is, that their sum is the weight of the body.

When the vertical line passing through the centre of gravity falls without the base, the body must upset; because, the reaction of the plane being in a vertical line different from that in which the weight is directed, those forces cannot equilibrate.

If the body is placed on an inclined plane, it must necessarily descend; the surfaces being perfectly smooth, *i. e.*

supposing no restraint arising from friction. For, if the weight of the body, which is a force applied at the centre of gravity, is resolved into two forces, one of them perpendicular, and the other parallel to the plane, the latter, not being opposed to the reaction, must produce its full effect in carrying the body down the plane.

If the perpendicular from the centre of gravity on the plane, falls within the base or surface of contact, the body shall descend by sliding. For if the weight of the body, which is a force applied at its centre of gravity, is resolved into two forces, of which one is parallel, and the other perpendicular to the plane: this last being directed to a point within the base, shall be equilibrated by the resistances: and as the remaining force, viz. that parallel to the plane passes through the centre of gravity, there will be no rotatory movement. See Art. 4.

From these considerations, it would appear that a sphere, cylinder, or a regular polyhedron should descend down an inclined plane by sliding, and not by rolling or tumbling; inasmuch as in these bodies, the perpendicular from the centre of gravity on the plane, always falls within the base, whatever be the inclination of the plane to the horizon. But in this, the force of friction is not considered. The effect of friction is to impede the motion of the parts of the body in contact with the plane. Accordingly, even though the perpendicular on the plane from the centre of gravity, should fall within the base, the body shall roll or tumble, whenever the force of friction becomes equal to that which is required to turn the body over the edge of its base. Hence, if the body is a sphere whose contact is reduced to a single point, the least imaginable friction will impart rotatory motion: but a polyhedron will require more or less, according to the extent of the base, measured from the foot of the perpendicular from the centre of gravity on the plane, and in the direction of the slope: and the same body that

slides, when opposed by a certain force of friction, may tumble if the friction is increased.

If the perpendicular on the inclined plane falls without the base, the body shall always descend by rolling or tumbling. But this rotatory movement may be in any direction, according to the position of the foot of the perpendicular, with respect to the base of the body or surface of contact. For the weight of the body resolved in a direction perpendicular to the plane, and the reaction of the plane, being parallel and contrary forces, must necessarily produce a rotatory movement, whose direction shall be determined by the position of the point, in which the plane is met by the perpendicular let fall on it from the centre of gravity.

It appears then, that a body placed on an inclined plane may tumble backwards, *i. e.* up the slope, even though it should lean forwards with respect to the vertical. This may be exemplified by a rod of inconsiderable thickness, when its direction is within the angle made between the perpendicular to the plane, and the perpendicular to the horizon.

6. The properties of the centre of gravity hitherto described, relate to a body suspended from a fixed point, or incumbent on a plane. There is another property of this centre which deserves to be noticed. It is that expressed by the following theorem:

"The content of a surface or solid of revolution is equal to the product of the generating line or plane, by the path described by its centre of gravity."

To prove this theorem relative to a surface of revolution, let DE be the line by whose motion the surface is generated, (Fig. 44,) and BC the axis of revolution: let s denote the whole line; ds any one of its elementary parts, and y its ordinate, or the perpendicular distance from the axis CB: also, let a denote the distance of its centre of gravity from

the line BC, and ω the angle described by the plane CBDE; observing that ω is an abstract number, viz. the quotient of the arc by its radius: so that the expression for the arc described by any point of the curve whose ordinate is y, is ωy.

This notation being understood, it will be seen that the conical surface described by any element, ds, is

$$\omega.y.ds.$$

and that the surface generated by the entire line, is

$$\int (\omega.y.ds).$$

or, because ω is the same for all the elements, it is

$$\omega \int (y.ds).$$

But the sum of the products, had by multiplying each element into its distance from the axis BC, is equal to the single product of their sum, s, into the distance of its centre of gravity from the axis, *i. e.*

$$\int y.ds = a.s.$$

wherefore,

$$\omega \int y.ds = \omega.a.s.$$

which, putting S for the generated surface, is

$$S = \omega.a.s.$$

i. e. the generated surface is equal to the product of the generating line, into the arc or path described by its centre of gravity.

To prove the proposition for a solid of revolution, let CBDE be the generating plane, and CB, as before, the axis of revolution. The plane being supposed to be resolved into elementary rectangles, by ordinates perpendicular to the axis CB; and dx, denoting the portion of the axis between any two consecutive ordinates, the expression for the elementary rectangle will be $y.dx$.

But the sum of the products had by multiplying each of

these elementary rectangles, into the distance of its centre of gravity from the axis, is equal to the single product of the area of the entire plane, into the distance of its centre of gravity from the same axis, *i. e.* putting A for the entire area CBED, and g for the distance of its centre of gravity from the axis CB, it will be

$$\int \frac{y^2}{2}\, dx = \text{A}.g.$$

Therefore,

$$\frac{\omega}{2} \cdot \int y^2 . dx = \text{A}.\omega.g.$$

The first member of this equation is the sum of the products of $y.dx.$ into $\frac{\omega . y}{2}$. or the sum of the products had by multiplying each elementary rectangle into half the arc described by the extremity of its ordinate, *i. e.* the sum of the portions of the solid generated by the several rectangles: and the second member is the area of the generating plane, multiplied into the arc or path described by its centre of gravity. Wherefore, putting V for the entire volume of the generated solid, it will be

$$\text{V} = \text{A}.\omega.g.$$

The whole of the surface or solid of revolution is expressed by replacing ω in these formulæ with 2π, the abstract number, which denotes the ratio of the periphery of a circle to its radius, *i. e.* the quotient of the former divided by the latter. Making this substitution, the surface of revolution is expressed by the equation

$$\text{S} = 2\pi . a . s.$$

and the solid of revolution by the equation

$$\text{V} = 2\pi . g . \text{A}.$$

If the revolving line or plane lies on each side of the axis, it is only the difference of the surfaces or solids generated

by the parts, at different sides of the axis, that is so expressed: which will immediately appear by considering, that in the statement relative to the equality of the product of an entire mass into the distance of its centre of gravity from a plane, and the sum of the products had by multiplying each of the partial masses into the distance of its centre of gravity from the same plane, those distances, if at opposite sides of the plane, are to be marked with opposite signs.

These equations give the magnitude of a surface or solid of revolution, when that of the generating line or plane is known, together with the distance of its centre of gravity from the axis of revolution. This method of quadrature or cubature is called the barocentric method. A few examples of its application are subjoined.

Example 1.—Let it be proposed to find the surface of a truncated cone, whose side is given, together with the radii of the circular bases.

This surface may be generated by the motion of the side of the cone round its axis: wherefore, putting r, r' for the radii of the circular terminations, we have

$$a = \frac{r + r'}{2}.$$

which value of a, being substituted in the general formula, there is

$$\text{S} = \pi (r + r') s.$$

Showing that the surface of the truncated cone is equal to the area of a circle, whose radius is a mean proportional between s and $r + r'$, *i. e.* between the side and the sum of the radii of the circular bases.

Example 2.—Let it be proposed to find the content of the solid, generated by the revolution of an isosceles triangle round a line passing through its vertex, and parallel to its base.

Let CBD be the triangle, and CM its altitude, (Fig. 45.)

and putting h for this altitude, and b for the base BD, we have

$$\text{A} = \frac{bh}{2} \qquad \text{and } g = \frac{2}{3} h.$$

Substituting for A and g these values in the general formula, there is

$$\text{V} = \frac{2}{3} \pi . h^2 . b.$$

But $\pi . h^2$ is the area of the circle whose radius is h, or CM, and $\pi . h^2 . b$ is the solid content of the cylinder standing on that base, and whose altitude is b, or BD. Wherefore, the solid is $\frac{2}{3}$ of that cylinder.

Example 3.—Let it be proposed to find the solid content of a ring.

This solid may be supposed to be generated by the movement of the circular section of the ring, its centre describing the periphery of another circle, to which its plane is every where perpendicular. Therefore, if r denotes the radius of the generating circle, and r' that of the circle described by its centre, the area of the generating plane will be

$$\pi . r^2.$$

and the path described by its centre of gravity will be

$$2 \pi . r'.$$

Accordingly, for the volume, we shall have

$$\text{V} = 2 \pi^2 . r^2 . r'.$$

If the axis, round which the generating circle revolves, is a tangent drawn to a point in its periphery, we shall have

$$r' = r.$$

and for the solid content,

$$\text{V} = 2 \pi^2 . r^3.$$

Example 4.—To find the content of the solid, generated by the revolution of a parabolic segment, round its chord.

Let the chord of the segment be perpendicular to the axis, and putting b for this chord, and h for the abscissa, the area of the generating plane is

$$\tfrac{2}{3}\, b.h.$$

Moreover, the distance of the centre of gravity from the vertex was found to be

$$\tfrac{3}{5}\, h.$$

and, therefore, the distance of the same from the axis of revolution, is

$$\tfrac{2}{5}\, h.$$

Wherefore, the path described by the centre of gravity, is

$$\tfrac{4}{5}.\pi.h.$$

giving for the volume

$$\text{V} = \tfrac{8}{15}.\pi b.h^2.$$

or putting for b, its value, viz. $2\sqrt{(p.h)}$. it is

$$\text{V} = \tfrac{16}{15}.\pi.p^{\frac{1}{2}}.h^{\frac{5}{2}}.$$

If the chord is oblique to the diameter to which it is applied, putting a for the angle of inclination, the area of the generating plane is

$$\tfrac{2}{3}\, b.h.\sin.a.$$

and the path described by the centre of gravity is

$$\tfrac{4}{5}.\pi.h.\sin.a.$$

whereby the equation becomes

$$\text{V} = \tfrac{16}{15}.\pi.p^{\frac{1}{2}}.h^{\frac{5}{2}}.\sin^2.a.$$

The same general formulæ may be applied to find the centre of gravity of the generating line or plane when the content of the generated surface or solid is known. Thus,

$$a = \frac{\text{S}}{2\pi.s}. \qquad g = \frac{\text{V}}{2\pi.\text{A}}.$$

Example 5.—The solid content of a sphere is

$$\tfrac{4}{3} \cdot \pi \cdot r^3.$$

and this solid is generated by the revolution of a semicircle round the diameter, by which it is terminated, whose area is

$$\frac{\pi r^2}{2}.$$

Wherefore, substituting these values of V and A in the last equation, we have for the centre of gravity of the semicircle,

$$g = \tfrac{4}{3} \cdot \frac{r}{\pi}.$$

The same as already found by the direct method.

In finding the distance of the centre of gravity of a line or surface from any assumed axis or plane, each element was multiplied into its distance from that axis or plane, and the sum of the products, thus obtained, was divided by the whole content of the figure: and in finding the content of the surface or solid generated by the revolution of that figure, the distance of the centre of gravity was multiplied into 2π, and into the content of the generating line or surface. It is evident then, that this division and subsequent multiplication, by the content of the generating figure, may be omitted; and that when the investigation is to be conducted by the aid of the integral calculus, it will be enough to take the sum of the products of each element into its distance from the axis, and multiply that integral by 2π.

SECTION VII.

OF THE MECHANIC POWERS.

1. THE use of mechanic instruments, as far as they are concerned in statics, is to enable us, by a force at our disposal, to counteract another force, to which it is not opposed in direction, and to economize the force employed for this purpose. The latter force is called the power, and the former the resistance; or simply the weight, when the resistance is a weight to be equilibrated: and it is evident, that the equilibrium shall be established, whenever the resultant of the power and resistance is directed against some fixed point, or immoveable obstacle: but then the equilibrium is not, properly speaking, between those forces, but between them, or their resultant, and the reaction of the fixed point or obstacle.

Such being the way in which the equilibrium is established by means of mechanic instruments, it is apparent that there may be the greatest disparity between the antagonist forces: and it is said that there is a mechanical advantage or disadvantage in the instrument, according as the resistance is greater or less than the power by which it is counteracted.

The more simple elements, into which machines are resolvable, are called mechanic powers. These may be classed according to their different structures and modes of application: but the classification is in a great measure arbitrary; the precise difference which constitutes a distinction of class not being generally agreed on. Accordingly, by some writers, they have been distributed under six, by

some under seven, and by others, under eight classes or heads. What renders the distribution of the mechanic powers still more uncertain is, that being only different means of effecting the composition or resolution of forces, the operation, in all, is reducible to the same principle. By each of them we are enabled to compound with the force to be counteracted, a new force called the power; so that the resultant of both shall be directed against the obstacle. The differences, then, of mechanic instruments, relate not to the principle of their operation, but to their structures and modes of application; and those who wish to treat the subject in the most simple manner, will be inclined to refer to the same class, such instruments as have an obvious resemblance in these respects. According to this rule, they may be classed under three heads, viz. the lever, the rope, the inclined plane. The first head comprising the balance, and the axle in the wheel; and the third head the wedge and the screw.

THE LEVER.

2. The lever is a bar capable of angular motion round a point called the fulcrum. To this, two forces are applied: that which is to be counteracted is named the resistance; and the force by which it is to be equilibrated, the power. The rectilinear distances of the points of application from the fulcrum are called the arms of the lever. Thus in (Fig. 46.) F is the fulcrum, round which the lever AB is at liberty to turn; P the power; and W the resistance or weight to be supported; and the right lines, FA, FB, drawn from the fulcrum to the points at which these forces are applied, the arms of the lever.

It is evident that the weight of the instrument itself is a force, which, if it does not pass on the obstacle, must come in aid of one or other of the forces to be opposed to each

other. But to deduce the consequences of certain conditions, these conditions must be contemplated apart, even from those with which they may be inseparably connected. Wherefore, in considering the relation between the forces, which are to equilibrate on a bar, with respect to angular motion, the bar is regarded as a line without flexibility or weight; and the centre of angular motion as a fixed point. When the weight of the instrument is to be taken into account, it is to be treated as a force applied at its centre of gravity in a vertical direction; and as making part of the power or resistance, according as it conspires with one or other of those forces.

3. For equilibrium, it is requisite and it is sufficient, that the power and resistance should have a single resultant directed through the fulcrum or fixed point. And this condition is equivalent to the three following: 1st. That the directions of power and resistance should be in the same plane which contains the fulcrum; for if the power and resistance did not act in the same plane, they could not have a single resultant; and if this plane did not contain the fulcrum, the resultant, which is always in the plane of the components, could not pass through this point. 2d. That the power and resistance should tend to turn the lever round the fulcrum in opposite directions; for if the directions of the forces meet at an angle, that of the resultant must lie within the same; *i. e.* the fulcrum must lie within this angle. And if the lines of direction are parallel, that of the resultant, and therefore also the fulcrum must lie between them or beyond them, according as the component forces are directed to the same or opposite sides of a line transverse to their directions. 3d. That the power and resistance should be reciprocally proportional to the perpendiculars let fall from the fulcrum on their lines of direction: for the components are reciprocally proportional to the perpendiculars let fall from any point taken in the line of the resultant, and the fulcrum is in

that line. Thus P, P′ representing the power and resistance, and p, p' the perpendiculars from the fulcrum on their directions, this condition requires the following relation:

$$\text{P} : \text{P}' :: p' : p. \quad \text{or, } \text{P}p = \text{P}'p'.$$

This relation may be otherwise expressed; for l, l' being the lengths of the arms, and θ, θ' the angles which they make with the directions of the forces, there is $p = l.\sin.\theta, p' = l'.\sin.\theta'$. whence

$$\text{P}.l.\sin.\theta = \text{P}'.l'.\sin.\theta'.$$

The product of a force into the perpendicular from the fulcrum on its direction, is called the moment of that force; and the three conditions of equilibrium may be briefly expressed, by saying, that "the moments of the forces should be equal and opposite."

If the fulcrum is not a point in the lever itself, but only a point of the surface of a body on which it may rest, it will be requisite to add to the conditions of equilibrium as stated above, that the resultant of the forces should be directed to the fixed point, in a line perpendicular to the surface at the contact.

4. Regarding the power as a force to be economized, it is usual to divide the lever into three kinds, according to the position of the fulcrum with respect to the power and resistance.

The first kind of lever is that in which the fulcrum lies between the power and the resistance or weight, as in (Fig. 47.)

The second is that in which the resistance or weight is applied between the power and fulcrum, as in (Fig. 48.)

And the third is that in which the power is applied between the resistance and fulcrum, as in (Fig. 49.)

In the first kind of lever, there is a mechanical advantage or disadvantage, according as the perpendicular from the fulcrum on the direction of the power is greater or less than that on the direction of the resistance or weight.

If the lever is straight, and the directions of the forces parallel, the perpendiculars from the fulcrum on the directions of the forces are proportional to the lengths of the arms. Therefore, in this case, the power acts at a mechanical advantage or disadvantage, according as it is applied to the longer or shorter arm of the lever.

In the other two kinds, the arms are always unequal; and, therefore, the lever being supposed to be straight, and the forces parallel in direction, there will be always a mechanical advantage in the second kind of lever, and a mechanical disadvantage in the third.

5. Hitherto two forces, only, were supposed to be applied to the lever; and in that case, it was required that they should act in the same plane; otherwise, they could not be compounded into a single resultant. And it was further required, that the plane of the forces should pass through the fulcrum; otherwise, their resultant could not pass through the fixed point. These things were contained in the general statement that the moments should be directly opposed. But when there are more than two forces applied to the lever, it is no longer requisite that the directions of the forces should be contained in the same plane: but then, it is required that the sum of the moments of those forces should be cypher, in each of the three coordinate planes passing through the fulcrum, *i. e.* that the three conditions of equilibrium should be satisfied as stated in Sect. V. Art. 2. relative to a system containing a fixed point.

If the fulcrum is a cylindrical axle, which allows the lever no liberty of movement, but in a plane perpendicular to that axle, the equilibrium is provided for, when the sum of the moments of the forces, reduced to that plane, is cypher: and if the lever only rests on a fixed point, the equilibrium is established, only when the forces are reducible to a single resultant directed against that point.

6. It is sometimes requisite to consider the strain on the

fulcrum, with a view to the safety of the instrument, or in estimating the effects of friction; and it is readily understood, that when the fulcrum is a fixed point, the strain on this point is the resultant of the forces applied to the arms of the lever. This resultant passes through the fulcrum, which may therefore be taken for its point of application, where it will reproduce its components. Accordingly, the strain on the fulcrum may be regarded as the resultant of the forces acting immediately at this point, in lines parallel to their proper directions. The magnitude of this resultant, or the charge on the fixed point, will depend not only on the magnitudes of the forces applied to the arms of the lever, but also on the angle contained between their lines of direction; the greater limit being the sum of those forces, which is the strain, when they are applied in parallel directions to a lever of the first kind: and the lesser limit being the difference of those forces, which is the strain, when they are applied in parallel directions to a lever of the second or third species.

7. All those mechanical instruments are to be regarded as levers, wherein the motion that ensues on a violation of equilibrium is circular. Thus, a hammer, when used for the purpose of drawing a nail, is a lever of the first kind; the power being applied at the end of the handle, the resistance at the claw where it grips the nail, and the fulcrum being the heel round which the instrument turns. Crow-bars, also, are levers of this kind, when the power and resistance move in opposite directions: as when the power, applied at one end, descends; whilst the weight, at the other end, ascends. But when both together ascend, or together descend, the weight is then between the fulcrum and the point of application of the power; and the instrument is, in this use of it, a lever of the second kind.

An oar of a boat is another example of the second kind of

lever: the fulcrum being in the water, the resistance at the row-lock, and the power applied by the hand of the rower.

A ladder to be raised against a wall, whilst one end remains on the ground, affords an example of the third kind of lever: the fulcrum being the end on the ground, round which it turns; the weight that of the ladder acting at the centre of gravity, and in a vertical direction; the power being applied by the hands of the labourer at some intermediate point. It is evident that the labourer applies his force with most advantage in the perpendicular to the ladder; and that the nearer the ladder approaches to the vertical position, the less is the perpendicular from the fulcrum on the direction of the weight; and, therefore, the less the force required to overcome it.

The limbs of animals, which are moved by muscular power, are also levers of the third kind: the fulcrum being at the head of the bone; and the muscles acting between this point and the centre of gravity of the limb, where the weight of its parts may be supposed to be concentred. This mode of action might seem to require a needless expenditure of animal force. But it is to be considered, how much is gained by this contrivance, in the lightness of the limb and the celerity of its movements: and whether these advantages are not wisely secured by an expenditure of force, which the enormous power of the muscles, in contracting, may so well afford.

A pole with a weight, carried by two men, may be viewed as a lever of the first kind, if the weight be regarded as the resistance of the fulcrum, and the bearers as acting against each other: or it may be viewed as a lever of the second kind, the force exerted by either of the bearers being regarded as the reaction of the fulcrum, and that exerted by the other as the power by which the weight is supported. Whilst the forces exerted are vertical, the sum is only equal to the weight to be carried, and the portions of this sup-

ported by the two men, are reciprocally proportional to their distances from its centre of gravity. But if the forces exerted by the bearers are not vertical, their directions must meet at some point of the vertical passing through the centre of gravity of the load; and as in the parallelogram of forces, the sum of the two contiguous sides is greater than the diagonal; so the sum of the forces exerted must exceed the weight to be supported: and they are to each other reciprocally as the perpendiculars, let fall from the centre of gravity of the load upon their lines of direction. Whilst the pole is horizontal, each of the bearers would suffer from this mode of exertion. For, let the pole be AB, (Fig. 50.); let the weight W be applied at the point o; and let P, P′ be the forces exerted by the bearers at A and B, in directions meeting the vertical passing through o, at the point c; and let Bd be perpendicular to Ac. Then,

$$\mathrm{P} = \mathrm{W} \times \frac{\mathrm{B}o}{\mathrm{B}d}.$$

But W and Bo, being constant quantities, the force P would vary inversely as Bd; *i. e.* inversely as the sine of the angle cAB. And in the same way it appears, that the force P′ varies inversely as the sine of the angle cBA. But when the pole is not horizontal, as when it is carried up or down a hill, the lowermost bearer must act, partly, by pushing or shoving; and the uppermost by pulling or drawing: and this mode of exertion being inconvenient to the latter, he will naturally be disposed to relieve himself at the expense of his fellow. To see to what extent he may diminish his burden, let AB be the pole in its inclined position, (Fig. 51.) and let Bd, Be be perpendicular to cA, co. Then,

$$\mathrm{P} = \mathrm{W} \times \frac{\mathrm{B}e}{\mathrm{B}d}.$$

in which Be, depending on the slope, is to be considered as a constant quantity. Wherefore, P shall vary inversely as Bd,

and shall, therefore, be least when this perpendicular is greatest, *i. e.* when Bd coincides with BA; or when Ac is perpendicular to AB. The force then exerted at A will be given by the equation

$$P = W \times \frac{Be}{AB}.$$

which is less than would be required in the horizontal position of the pole, in the ratio of Be to Bo; *i. e.* in the ratio of the cosine of the angle of elevation to radius. The additional exertion required of the lowermost bearer will be greater than that from which the uppermost thus relieves himself; inasmuch as, in this case, the sum of the forces exceeds the weight to be supported.

8. Several levers may be combined together; and then the action of any one of them on the next that follows, is the power by which the latter is worked. Thus, ABCD (Fig. 52.) represents a combination of three levers of the first kind; F', F'', F''', being the fulcra. The manner in which the action is conveyed from one extremity to the other, is as follows: A, the extremity of the first lever, being depressed by the power, the other end of that lever is raised; and together with it B, the end of the second lever, which presses on it. This elevation of the nearer end of the second lever produces a depression of its remoter end; and, therefore, of C, the nearer end of the third lever, by which its remoter end, D, is raised.

The mechanical advantage of this combination is readily understood. The power applied at A is to the force which it equilibrates at B, as $F'B$ to $F'A$. The force at B is to that which it equilibrates at C, as $F''C$ to $F''B$. And the force at C is to that which it equilibrates at D, as $F'''D$ to $F'''C$. Wherefore, the power applied at A is to the weight which it will sustain at D, as $F'B \times F''C \times F'''D$, to $F'A \times F''B \times F'''C$; *i. e.* as the product of the several powers to that of the several weights,

which they would balance in the levers taken separately. Thus, if

$$F'A : F'B :: 3 : 2. \quad F''B : F''C :: 7 : 3. \quad F'''C : F'''D :: 5 : 2.$$

Then,

$$F'B \times F''C \times F'''D : F'A \times F''B \times F'''C :: 2 \times 3 \times 2 : 3 \times 7 \times 5.$$

or as 4 to 35. So that a force equivalent to a weight of four pounds at A, shall sustain a weight of thirty-five pounds at D. And if the power at A were made greater than this, the weight at D would be lifted.

Such are the conditions of equilibrium in the lever, and such, in general, is the manner in which this instrument is applied. But there are certain instruments of this class, which, from some peculiarity in their uses or structure, would demand a more particular consideration.

9. The balance is a lever, applied to ascertain the weight of a body.

There are five different ways in which the equilibrium may be established: and these furnish so many different indications of the weight of the body under examination. 1st, By adjusting the weight of the counterpoise acting at a given distance from the fulcrum or axle. 2d, By changing its distance from the fulcrum. 3d, By changing that of the body to be weighed from the same point. 4th, By shifting the fulcrum. The fifth indication is the inclination of the beam when it composes itself.

Of these several indications, the first is most generally resorted to; recommended, no doubt, by its more extensive application; as also, by its superior accuracy. For this reason, the adjustments to be attended to in a balance constructed on this principle, shall be more particularly considered.

THE BALANCE FOR EQUAL WEIGHTS.

10. In this balance the body whose weight is to be ascertained and its counterpoise are placed in dishes, or otherwise suspended from the extremities of the beam, and are therefore to be regarded as acting immediately at the points of suspension; and the fulcrum on which the beam turns is situated between those points: the instrument is evidently a lever of the first kind, whose arms are the right lines drawn from the fulcrum to the points of suspension. Now, as the equilibrium is to be established between equal weights, it is plainly requisite that the arms should be of equal lengths; and at first view this may appear sufficient. But as this kind of balance is frequently employed in cases where the most scrupulous accuracy is desired, it is necessary to consider more particularly the properties which are required, and the adjustments to be attended to.

For the indication of the equality of the appended weights, it is necessary that the beam should compose itself in some certain position; and the horizontal position is obviously the most convenient. Wherefore, the property first required in this balance is, that it should rest in none but the horizontal position when loaded with any equal weights, and therefore also when unloaded. The second is, that it should quickly right itself when disturbed from that position: this is called its stability. The third property required is that it should indicate a small difference of weights by its deviation from the horizontal position. This is called its sensibility. It now remains to be shown how these perfections are to be attained.

11. The balance cannot have the first of these properties unless the common centre of gravity of the beam and equal weights lies beneath the axle; for were it at this point, the balance would rest indifferently in any position; and were it

above the axle, the smallest deviation from the vertical passing through this point would cause the balance to upset.

It is necessary then that the common centre of gravity of the beam and equal weights should lie beneath the axle; but this is not enough. In order that the equality of the weights should be indicated by a certain position of the beam, it is necessary that the right line drawn from the fulcrum to the centre of gravity of the unloaded beam should bisect the line connecting the points of suspension. To show this, let A*o*B be the balance, (Fig. 53.) *o* its fulcrum, *b* the point at which the line AB is bisected. This point is the centre of gravity of the equal weights, where they may be supposed to be concentred. Let *c* be the centre of gravity of the unloaded beam, then the common centre of gravity of the beam and equal weights shall be in some point of the line *bc*, as at *x*, dividing the line *bc* into segments *xb*, *xc*, which are reciprocally proportional to the masses concentred at *b* and *c*, and therefore changing its place in the line *cb* with every change in the magnitude of the load. Hence it appears, that if the line *bc* does not pass through the fulcrum *o*, the line *ox* shall make different angles with the line AB; and since the line *ox* is necessarily vertical when the balance is at rest, it follows that the position of AB shall change with every change in the magnitude of the load.

It is plain that when the line *bc* passes through *o*, the position of equilibrium for equal weights is independent of their magnitude.

But it is not sufficient that the equality of the weights should be indicated by the same unvaried position of the beam, that position must be horizontal, and the former condition being satisfied, this last is provided for by making the arms of equal lengths. For in the last figure the line *ob* being vertical when the balance is at rest, if AB is horizontal the angles at *b* are equal, and therefore the lines *o*A, *o*B are also equal: and conversely, if these lines are equal, the

angles at b must be equal, *i. e.* right, and therefore the line AB horizontal.

If the arms are of unequal lengths, the lines ob, bA (Fig. 54.) being respectively equal to ob, Bb as before, and oA, oB being unequal, the angles at b are unequal, and ob being vertical, AB shall be no longer horizontal, but inclined.

This suggests a ready mode of trying whether the condition relative to the lengths of the arms is fulfilled. For it is requisite only to interchange the equilibrating weights; and if the equilibrium still subsists, the arms are precisely of equal lengths. Were it otherwise, the weight which had acted by the shorter arm, being the greater weight, now that it acts by the longer arm, must necessarily preponderate.

A balance which does not stand this test, though it may rest in the horizontal position when unloaded, is a deceitful balance: and the commodity to be dealt out, if placed at the end of the longer arm, is deficient in weight; bearing to that of the counterpoise, the ratio of the shorter to the longer distance. If placed in the dish at the end of the shorter arm, and again counterpoised, the weight of the new counterpoise shall bear to that of the commodity the same ratio, as the latter to that of the first counterpoise: wherefore, the true weight is a geometrical mean between those by which it is counterpoised, when weighed in the opposite scales, and may be ascertained accordingly. Thus, putting P for the weight of the commodity; l for the longer line BC; and l' for the shorter AC; and when P is appended by the longer arm, let A be its counterpoise: we shall then have

$$\mathrm{P}.l = \mathrm{A}.l'.$$

Again, when appended by the shorter arm, let B be its counterpoise, and the equation will be

$$\mathrm{P}.l' = \mathrm{B}.l.$$

To exterminate l, l', let these equations be multiplied, and we get

$$P^2 = A.B. \qquad \text{or } P = \sqrt{(A.B)}.$$

A still more easy way of using such a balance would be, after counterpoising the body, to take it out of the scale, and restore the equilibrium by a weight. For those weights are necessarily equal, which equilibrate with the same counterpoise in the same circumstances.

In order that the lengths of the arms should remain unvaried, in all positions of the beam, it is requisite that the bodies, weighed against each other, should not be applied to the arms in any considerable part of their surface; otherwise the vertical, passing through the centre of gravity of the body, shall meet the arm in a point whose distance from the axle is variable with the inclination of the beam; and the effective length of the arm shall be changed accordingly. To render the points of application invariable, the ends of the arms are formed into rings, and bent over at right angles. Into these are inserted the hooks, which are to carry the dishes or weights; and the bearings being reduced to knife edges, the points of suspension are invariably the same.

12. Another of the perfections, which were said to be required in a balance, is stability. This relates to the celerity with which it rights itself, when disturbed from the horizontal position; and, therefore, it depends on the force of restitution, which may be thus estimated.

Let AB be the line connecting the points of suspension; (Fig. 55.) W the weight of the unloaded balance; P one of the equal weights; g the common centre of gravity of the whole; and o the axle. Then, putting a for og, and θ for the angular deviation of og from the vertical, or of AB from the horizontal position, the force of restitution will be

$$(2\,\text{P} + \text{W})\, a \,.\, \sin.\, \theta.$$

i. e. if θ is of given magnitude, the force of restitution will be as

$$(2\,\text{P} + \text{W})\, a.$$

13. The third of the perfections required in a balance is its sensibility; which is estimated by the difference of the weights required to produce a given inclination of the beam in relation to their sum; the sensibility being the greater, as this difference is less, in relation to the entire load.

In order to see on what this property depends, let w be the additional weight thrown into the scale appended from A, (Fig. 55.) The effect of this will be, to remove the common centre of gravity of the balance and weights from g towards A, as to g'. The balance will now compose itself, so that og' shall be vertical; and the deviation of the beam from the horizontal position is measured by the angle $g'og$.

Now, for the relation between this angle and the quantity $\frac{2\,\text{P} + \text{W}}{w}$, there is in the bent lever Aog, the weight w, at A, in equilibrio with $2\,\text{P} + \text{W}$, at g. Wherefore, putting L for oA, there will be

$$w \,.\, \text{L} \,.\, \sin.\, \text{ang. A}og' = (2\,\text{P} + \text{W})\, a \,.\, \sin.\, \theta.$$

But when AB passes through the axle, the angle Aog is right, and Aog' is the complement of θ, which gives

$$\tan.\, \theta = \frac{w \,.\, \text{L}}{(2\,\text{P} + \text{W})\, a}.$$

Though the line AB should not pass through the axle, yet the angle oAB is always exceedingly small; and, therefore, Aog nearly a right angle. So that the error will not be, in any case, considerable, if Aog' is treated as the complement of θ. Accordingly, the preceding equation will serve generally for the straight beam. If then, θ is of given magnitude,

$\frac{2\text{P}+\text{W}}{w}$, shall vary as $\frac{\text{L}}{a}$: which shews that the sensibility is greater, as the length of the arm is increased, and as the distance of the common centre of gravity from the axle is diminished.

14. If the centre of gravity of the unloaded beam is in the line connecting the points of suspension, and at its middle point, as required Art. 11. the length of the line a, shall be independent of the magnitude of the appended weights; and $\frac{\text{L}}{a}$, which is the measure of the sensibility, will not be affected by any change of load.

But if the line connecting the points of suspension does not pass through the centre of gravity of the balance itself; it is evident, that as the magnitude of the appended weights is increased or diminished, the common centre of gravity of beam and weights shall approach this line or recede from it accordingly: and that a being consequently a variable quantity, the sensibility and stability of the balance shall vary with the load in a manner easily understood.

Thus, if the line connecting the points of suspension falls below the centre of gravity of the unloaded balance, the sensibility shall be diminished, and the stability increased, as the weights are increased. And if the same line falls above the centre of gravity of the unloaded beam, between it and the axle, the effect will be the contrary. And finally, if the same line falls above the axle, the balance will serve only for weights below a certain limit. This limit of 2 P depends on W, the weight of the balance, and on the distances of the axle from the centre of gravity of the unloaded balance, and from the line connecting the points of suspension. For let o be the axle; g the centre of gravity of the unloaded balance; and let the line go meet the line AB above the point o, as at c, (Fig. 56.) This point c is the centre of gravity of 2 P,

and it is evident that as this weight is increased in relation to W, the common centre of gravity of the whole shall ascend from g towards c; and that when it arrives at o, the balance becomes useless. In that case we have

$$2\text{P} \times oc = \text{W} \times og.$$

and therefore,

$$2\text{P} = \text{W} \times \frac{og}{oc}.$$

At this limit the balance shall rest indifferently in any position. And if the weights are further increased, the common centre of gravity shall be raised above the axle; and the balance, on receiving the smallest inclination, shall upset. If the point g lies above, and the point c below the axle, the balance will serve only for weights above a certain limit. This limit being expressed as before.

The qualities of stability and sensibility are not to be combined together: and if the balance is improved in one respect, it must be injured in the other; as will more fully appear from the theory of pendulums. Therefore, nice balances are often provided with an adjusting strew; by which the distance of the common centre of gravity from the axle may be varied at pleasure, according to the degree of sensibility required, by the uses to which the balance is to be applied.

Various contrivances have been adopted for lessening the weight of the beam without prejudice to its strength or stiffness. In ordinary cases, this is supposed to be sufficiently provided for by the shape and material of the beam. This is made of steel. Its thickness is far less than its depth; and this latter decreases from the axle to the extremities. The sensibility of the balance is also affected by the friction between the axle and its supports. This is diminished, by attending to the material and polish of those parts, and by the form of the axle, which is that of a knife edge.

THE STEEL YARD.

15. The balance already examined is, perhaps of all, the most accurate. Other constructions are, however, frequently employed, recommended by their simplicity; but chiefly by the promptitude of their indications, which is often of more value than extreme precision.

The steel yard is a beam with arms very unequal in length. The commodity to be weighed is suspended by a hook at the end of the shorter arm, and is counterpoised by a sliding weight, hung by a steel edge on the longer arm. This arm is graduated; and the weight of the commodity is indicated by the division at which the counterpoise is placed, when the equilibrium is established. This, as well as the balance before described, is a lever of the first kind: and it is evident, that if at the first division, the counterpoise is in equilibrio with a certain weight, appended at the shorter arm; it shall balance twice that weight, when removed to the second; thrice when removed to the third division, and so forth. If, for example, the length of the shorter arm is two inches, and the counterpoise a weight of two pounds; then, when placed at the distance of two inches from the axle, it shall balance a weight of two pounds, hanging from the shorter arm; at the distance of four inches, one of four pounds; and at six inches, one of six pounds; every inch of the scale, in this case, corresponding to a pound weight. These divisions may be conveniently subdivided into parts of the eighth of an inch; and then the same balance will shew a difference of two ounces.

16. It is not requisite that the beam should compose itself in the horizontal position when unloaded, and it is therefore commonly made to hang vertically, as the most convenient position. But in that case, the zero of the scale

is not at the axle: for the moment of the beam conspires with that of the commodity or of the counterpoise, according as it is the shorter or the longer arm of the unloaded balance that preponderates: and for equilibrium, the equation will not be between the moments of the commodity and counterpoise; but between the moment of one of these, and that of the other increased by the moment of the beam. Therefore, it cannot be said that the weights of the commodity and counterpoise are inversely as their distances from the axle, *i. e.* the zero of the scale is not at this point.

To find the point from which the divisions should commence, let P be the weight of the commodity; p that of the counterpoise; and H the point where it is to be placed to render the beam horizontal, when otherwise unloaded. (Fig. 57.) It is plain that $p \times \mathrm{H}o$ is the moment of the beam. Therefore, when the equilibrium is established in the loaded beam, the weight P being at A, and the counterpoise at K, the equation is

$$\mathrm{P} \times \mathrm{A}o = p \times \mathrm{H}o + p \times \mathrm{K}o = p \times \mathrm{HK}.$$

Accordingly, the zero of the scale is at H; and if from this point, the portions HB, BC, CD, &c. are measured off, each equal to Ao, we shall have

$$\mathrm{P} = p, \qquad \mathrm{P} = 2p, \qquad \mathrm{P} = 3p, \ \&c.$$

as the counterpoise is placed at B, C, D, &c.

17. There are two limitations to the use of this balance, viz. the shortness of its range, and the coarseness of its indications.

When the counterpoise is carried to the end of the scale, it is in equilibrio with a weight greater, only, in the ratio of the longer to the shorter arm: so that if the lengths are as ten to one, and the counterpoise a weight of two pounds, the balance shall weigh only to twenty pounds.

This imperfection may be remedied, in part, by changing the counterpoise, according to the magnitude of the

weight to be examined. In this way, the value of each division is changed with that of the counterpoise.

Another method is that of furnishing the shorter arm with a second hook. If this hangs at half the distance of the former, the shorter arm is reduced to one-half its former length, and the value of each of the divisions on the longer arm is doubled.

With respect to the second limitation, it is to be observed, that the accuracy of weighing consists in the smallness of the difference of weights, which the balance will indicate, in relation to the load it carries. If the smallest divisions indicate ounces; then, unity divided by the number of ounces in the weight of the body examined, shall denote the degree of accuracy to which the weight is ascertained. This is the same fraction, as the smallest division of the scale, divided by the distance at which the counterpoise is placed to equilibrate the body weighed. So that if the counterpoise stands at fifty inches, and the smallest divisions are $\frac{1}{10}$ inch, the nicest indication will be the $\frac{1}{500}$ of the weight. This is far short of the nicety of the common balance last described. A well-made balance of that construction would shew a difference of the millionth part of the entire weight. So much more accurate are the ordinary methods of weighing than of measuring.

The want of a minuter division of the scale may be partly supplied by a second and smaller sliding weight. If, for example, the lighter be the $\frac{1}{10}$ of the heavier; then, the motion of the lighter through one division, produces the same change of momentum, as would the motion of the heavier, through $\frac{1}{10}$ of a division: and this simple contrivance answers the same purpose as a division of the scale ten times as minute. In using the two counterpoises, the heavier should be placed at the division nearest to equilibrium, and the defect is shewn by the division to which the smaller is applied, to render the equilibrium perfect. Thus, if the

divisions with the greater counterpoise denote ounces, and the weight of the body is found by it to be greater than 10lb. 6oz. but less than 10lb. 7oz.; then, if the lighter be $\frac{1}{10}$ of the heavier, and that the equilibrium is made perfect by advancing it to the third division, the weight of the body under examination is 10lb. 6.3oz.

In making a comparison of this with the ordinary balance, it should be observed that the load on the fulcrum exceeds the weight of the body under examination only by the weight of the counterpoise; which is, ordinarily, much less than it: whereas, the common balance is loaded with twice the weight of the commodity. This is a trifling advantage, and it is far more than counterbalanced by the disadvantages already mentioned. However, in cases where expedition is more desirable than extreme precision, this instrument is highly valuable.

THE BENT LEVER BALANCE.

18. The balance which gives the readiest indications is the bent lever balance; such as the corn balance, or the yarn balance, represented in (Fig. 58.) We have only to place the matter to be weighed in its scale; the balance produces its own equilibrium, and this most speedily, because of its great stability: the index, without any further trouble, declares the weight.

To ascertain the principles on which it should be graduated, let the balance be that represented in (Fig. 58. No. 1.) the arm which carries the scale being at right angles with the index, and this index being vertical before the weight is introduced; and, therefore, passing through the common centre of gravity of the balance with its scale and given counterpoise. Then, if a weight is placed in the scale, the common centre of gravity of the instrument and weight shall take the position vertically beneath the axle; the index as-

cending through a certain arc θ; and co.θ will become the angle, made between the arm which carries the scale and the vertical. Let l be the length of this arm; a the distance of the centre of gravity of the instrument from the axle; P the weight of the commodity in the scale; and p that of the instrument. Then, we shall have

$$\text{P}.l.\cos.\theta = p.a.\sin.\theta.$$

and therefore,

$$\text{P} = \frac{p.a.\tan.\theta}{l}.$$

But $\frac{p.a}{l}$ is a given quantity; wherefore, P varies as tan.θ, *i.e.* as the tangent of the angle through which the index has moved. If, then, by placing in the scale a weight of one pound, the index is moved through a certain angle; by a weight of two pounds, it shall be moved through an angle whose tangent is twice; and by a weight of three pounds, through one whose tangent is thrice as great; and so forth. Accordingly, to graduate the arc, beginning at the point where the index rests when the balance is unloaded, it is only requisite to find the arc through which the index is carried by a known weight; and putting unity for its tangent, to measure the other arcs from the same point to the several points of division, such, that their tangents shall be represented by the numbers 2, 3, 4, &c. Thus, let OA (Fig. 59.) be the arm to which the weight is to be appended; this being supposed to be horizontal when unloaded, and the index in the vertical *ov*. Then a known weight, *e. g.* of one pound being appended, let the arm take the position OA′, and the index that of *oa*. In the tangent *vs*, taking *ab*, *bc*, *cd*, &c. each equal to *va*, the lines *ao*, *bo*, *co*, *do*, &c. drawn to the centre shall mark the divisions on the arch.

In this explanation it was supposed that the arm which

carries the dish was horizontal, and the index vertical, when the instrument was unloaded, *i. e.* that it passed through the centre of gravity of the unloaded instrument. These conditions are not necessary. The arm and the index may have any positions. It is enough that the zero of the scale should be at the point marked by the index when the balance is unloaded: for, being fixed to the bent lever, it shall partake of its angular motion; and, therefore, describe the same angle as its centre of gravity. Accordingly, the index itself is commonly made to serve for the counterpoise: but if it is desired that the weights in the scale should be as the tangents of the arches described, it will be requisite that the arm which carries the dish, should be at right angles to the line drawn from the axle to the centre of gravity of the instrument, *i. e.* that the arm should be horizontal when the scale is empty.

A quadrantal arc would give an infinite range; the tangent of a right angle being infinite. But as the arches, whose tangents are equidifferent, increase by differences which rapidly diminish, and as the weight which the instrument would carry is limited; it is evident, that a graduated arc, much less than a quadrant, will serve for all practical purposes.

THE WEIGHING MACHINE.

19. The machine, the best that has been hitherto contrived for weighing heavy burdens, is that called the weighing machine. The scale is a platform, on which the cart or dray is driven; and the load, though of many hundreds weight, is balanced by a few pounds.

The equilibrium between weights so very unequal, is produced by means of a compound lever, as shewn in (Fig. 60.) where AKB is a lever, having two fulcra at A and B. DIC is another lever precisely similar, having its two

fulcra at D and C. At E, F, G, H, are upright pins, on which the platform rests; and by these the load on the platform is made to press on the levers AEB, DEC. These, then, are levers of the second order: and the pins being equidistant from the fulcra, the weight sustained by any one of them, as by the former at E, is to the pressure produced by it at K in the ratio of AK, to AE. The pressure thus reduced is communicated to a lever, ML of the second order, whose fulcrum is at L, and would be counteracted by a force at M, less in the ratio of LK to LM. So that if $AE = \frac{AK}{7}$, and $LK = \frac{LM}{8}$, a hundred weight on the platform would be counterpoised by a weight of two pounds acting at M. To diminish, yet further, the weight requisite for equilibrium, the point M is connected with the shorter arm of a balance, whereby the counterpoise is lessened in the ratio of the shorter to the longer arm. This balance may be a steel yard; and then the load is indicated by the division to which the sliding weight is brought, in order to produce equilibrium. Or else, the longer arm may be furnished with a dish appended to its extreme point; and then the load is ascertained by the weight in the dish. If the longer arm is only four times that of the shorter, it is evident that in the example given above, every hundred weight on the platform would be counterpoised by a weight of half a pound.

THE DANISH BALANCE.

20. The most portable balance, as well as the most simple in its structure, is that in which the equilibrium is established by the movement of the fulcrum: such is the balance AB, (Fig. 61.) called the Danish balance. This is simply a rod with a knob at one end to serve as a counter-

poise, and a hook at the other end to carry the commodity to be weighed. The equilibrium is established by shifting the place of the loop, by which the whole is sustained; and the weight is indicated by the mark at which the loop is then placed.

The principle of the graduation is easily understood. The weight of the rod, including the fixed counterpoise A, may be supposed to be applied at C, the common centre of gravity; and before the load is appended, the instrument would be supported horizontally by the loop at this point. But if a weight is hung from the hook, the equilibrium is established by shifting the loop to the common centre of gravity of the weight and balance, *i. e.* to a distance from C towards B, depending on the magnitude of the appended weight. Therefore, the part of the rod between its centre of gravity, and the end which carries the weight, is to be graduated, so that the divisions, reckoning from the former point, shall correspond to weights increasing by some common difference. These divisions are of unequal magnitudes; but they are readily calculated.

For let p denote the weight of the instrument, including the hook and fixed counterpoise; P the appended weight; and let E be the point to which the loop is to be shifted for equilibrium. Then putting a, for CB; x, for CE; we have

$$x : a - x :: \mathrm{P} : p. \qquad \text{and } x : a :: \mathrm{P} : \mathrm{P} + p.$$

Wherefore,

$$x = \frac{\mathrm{P}}{\mathrm{P}+p} \cdot a.$$

Which, as a and p are known quantities, gives the value of x for every assumed value of P. Thus, if p were eight ounces, and it were required to weigh to one ounce: there would be, for the first division, or the distance of the first mark from C,

$$x = \frac{1}{1+8} \cdot a = \frac{a}{9}.$$

For the second mark,

$$x = \frac{2}{2+8} \cdot a = \frac{a}{5}.$$

For the third,

$$x = \frac{3}{3+8} \cdot a = \frac{3}{11} \cdot a.$$

For the fourth,

$$x = \frac{4}{12} \cdot a = \frac{a}{3}.$$

and so forth.

THE AXLE IN THE WHEEL.

21. The lever is evidently an instrument of great power in overcoming a resistance; or even in communicating motion to large masses. But it would seem that the space through which they could be moved by it must be small; and it might be supposed that its usefulness would be limited by this condition.

There are, however, various methods of converting the reciprocating motion of the lever, into one that shall be rotatory or progressive.

The most simple contrivance for applying the principle of the lever to the production of progressive motion, is that of the axle in the wheel. This is an instrument consisting of a wheel fixed on a cylinder, which turns along with it. The moving power is made to act at the circumference of the wheel by a strap or cord; by coggs or teeth; or some other equivalent contrivance: and the weight is, in like manner, applied at the surface of the cylinder or axle.

The axle in the wheel is therefore a lever of the first kind; the arm by which the power acts being the radius of the wheel, and that to which the weight is applied, being the radius of the axle. Moreover, as the forces act in the di-

rection of the tangents to the wheel and axle, their directions are perpendicular to the arms of the lever: wherefore putting R, r, for the radii, P for the power, and W for the weight, there is for equilibrium $\text{PR} = \text{W}r$. or $\frac{\text{P}}{\text{W}} = \frac{r}{\text{R}}$.

But the axle in the wheel is a broken lever: the radii of the wheel and cylinder, which are the arms, being connected by the intervening portion of the axis, which meets them at right angles. And it may be supposed that the legitimacy of this application of the theory of the straight lever, to one thus distinguished from it, would require to be demonstrated. This may be done in several ways, of which the following is, perhaps, as simple as can be desired.

Let the force P, acting in the direction MN, be transferred to C, the centre of the wheel, (Fig. 62.) by applying at this point two forces, each equal to P, acting in the opposite directions, CE, CF, parallel to MN. This, which produces no change in the system, gives a force $+$ P, pressing on the axis of the cylinder at C; and a pair of forces $+$ P, $-$P, acting in the directions MN, CF, whose moment is P.R. The force, which is the weight W, being treated in the same way, gives a force $+$ W pressing on the axis at c, and a pair whose moment is W.r. The pressures on the axis, which is a fixed line, are necessarily equilibrated by its resistance: wherefore, the only condition required for equilibrium is, that the resultant of the moments should be cypher. And as the planes of the moments are parallel, this condition requires that their sum should be cypher, (Sect. III. Art. 2.) *i. e.* $\text{P.R} - \text{W}.r = 0$. or $\frac{\text{P}}{\text{W}} = \frac{r}{\text{R}}$ as before.

In this statement no account has been taken of the thickness of the rope. If, for greater accuracy, this is to be considered, the forces may be supposed to be applied at its axis; and then, to the radius of the wheel or cylinder is to

be added half the thickness of the rope by which it is enveloped. Thus, let the thickness of the rope which passes round the wheel be $2t$; and that which passes round the cylinder $2t'$; then for equilibrium the equation will be

$$\frac{\text{P}}{\text{W}}=\frac{r+t'}{\text{R}+t}.$$

22. With respect to the pressure on each of the points of support, it is to be observed, that this can arise only from the force equilibrated by the resistance of the axis, *i. e.* from the forces P and W transferred to this line; and from the weight of the engine itself, acting at its centre of gravity, which is in the same line. The pressure made by each of these forces is to be estimated as for a system secured by two fixed points: *i. e.* each force is to be resolved into two parallel forces, acting at those points. The total pressure on each point of support is then found, by compounding the forces exerted at that point. Thus, putting a for Ss, the distance between the points of support; m for CS; the pressures produced by P, at S and s, will be

$$\text{P}.\frac{a-m}{a}. \qquad \text{P}.\frac{m}{a}.$$

In the same manner, putting n for oS, the distance of the weight from the same point of support, the pressures made by W, at S and s, shall be

$$\text{W}.\frac{a-n}{a}. \qquad \text{W}.\frac{n}{a}.$$

Finally, let w denote the weight of the instrument itself; d the distance of its centre of gravity from S; the pressures made by this force at the same points, will be

$$w.\frac{a-d}{a}. \qquad w.\frac{d}{a}.$$

Wherefore, the total pressure at S is the resultant of the forces, $\text{P}.\frac{a-m}{a}$, and $\frac{\text{W}(a-n)+w(a-d)}{a}$, of which, the former

acts in a direction parallel to P, and the latter in the vertical. And the pressure at s is the resultant of the forces $\text{P}.\frac{m}{a}$, and $\frac{\text{W}.n+w.d}{a}$, acting in directions respectively parallel to the former.

If the direction of the force P is vertically downwards, the pressure at S will be

$$\frac{\text{P}(a-m)+\text{W}(a-n)+w(a-d)}{a}.$$

and that at s, will be

$$\frac{\text{P}.m+\text{W}.n+w.d}{a}.$$

The power, instead of being applied at the circumference of a wheel, is frequently applied to a lever inserted in the cylinder, as in the case of the windlass or capstan of a ship; and then the moment of the power is the product of this force, into the radius of the circle described by the point to which it is applied. When the resistance is to be overcome by manual force, this is most commonly applied to a winch handle, which is a lever consisting of three parts, whereof two are parallel; and these are joined by another part at right angles. One of the parallel divisions is a prolongation of the axis of the cylinder, and to the other the hand is applied. The leverage of the power is to be estimated by the length of the intermediate division. This is exhibited by CDEF, (Fig. 63.) The part EF, to which the hand is applied, is usually cased in a hollow cylinder which turns on it; and by this contrivance the friction is transferred from the hand to the interior surface of the hollow cylinder.

23. If several weights are applied to the axle, the equilibrium is established, when the product of the power into the radius of the wheel is equal to the sum of the products, had by multiplying each of the weights into half the thick-

ness of the axle, taken at the point where it acts. If, however, any of the weights acts in the same direction as the power, with respect to rotatory motion, its moment must be added to that of the power, or subducted from those of the remaining weights. Thus, if the weights are W, W′, W″, acting by the radii r, r', r''; and if W″ tends to turn the instrument in the same direction as the power, W, W′ tending to turn it in the opposite direction, the equation for equilibrium will be

$$P.R + W''.r'' = W.r + W'.r'.$$

or,

$$P.R = W.r + W'.r' - W''.r''.$$

In this way the labour is diminished, when the work to be performed consists of two opposite motions. Thus, in drawing up ore out of mine-shafts, two buckets are attached to the same axle or cylinder: the ropes, by which they hang, being wound in opposite directions. As the loaded bucket ascends, the empty one descends; whereby its moment is added to that of the power, or subducted from that of the resistance to be overcome.

Even when the work to be performed is of one kind only, the same principle has been applied to diminish the labour. To understand the value of the contrivance by which this is effected, it is to be observed, that the efficacy of the instrument is increased by enlarging the diameter of the wheel, or by diminishing that of the axle: that the former of these methods is limited by the size, which the engine cannot conveniently exceed; and the latter, by the necessity of leaving sufficient strength in the parts; and also, by the waste of power occasioned by the rigidity of the rope, when the curvature which it is to receive is too quick, *i. e.* when the axle round which it is to be coiled is too slender. But without incurring either of these inconveniences, the power of the instrument is increased by the following contrivance. The

axle consists of two parts, of different thickness; and the rope, which carries the weight, is attached by its ends to these parts, being wound round them in opposite directions; so that whilst it coils on one of them, it winds off the other; and the weight is suspended by a ring or pully from the rope where it hangs double. Each part of the rope is then strained by half the weight; and the moment of this strain is had by multiplying it into the radius of the axle to which that part of the rope is applied. The moment ef the weight is therefore the difference of these moments, or $\frac{1}{2}$ W $(r-r')$, and the equation for equilibrium is

$$\text{P.R} = \text{W}\,\frac{(r-r')}{2}.$$

This is evidently the same thing as if the axle were reduced to a radius equal to half the difference of the radii of the parts, whilst the strength in those parts is that due to r and r'. Thus, if the radius of the winch handle is supposed to be 20 inches; and the radii of the axle in its two parts, 5 and 6 inches; the power of the instrument, estimated by the ratio of the equilibrating forces, is,

$$\frac{\text{W}}{\text{P}} = 40.$$

So that an exertion of muscular strength, equal to half a hundred weight, would be put in equilibrio with a weight of one ton. To propose obtaining the same mechanical advantage, by reducing the diameter of a single axle or cylinder, its diameter should not exceed one inch, and a rope, sufficient to carry a ton, could never be coiled round such a cylinder.

24. But the usual method of increasing the mechanical effect in instruments of this kind, is by combining two or more of them in the same engine. One wheel may be made to drive another, by a band or strap passing round the axle of the driving wheel, and round the circumference of that

to be driven by it. But the method practised where much force is employed, is by furnishing the wheels to be driven with cogs or teeth; and the axle by which the motion is to be communicated, with a smaller wheel, similarly indented. The smaller wheels are called pinions; and their teeth, leaves: and as any one leaf of the pinion parts from a tooth of the wheel it drives, the next leaf of the pinion comes into contact with the next tooth of the wheel. Accordingly, the intervals between the teeth that so work together, must be equal; and the number of those in the pinion must be to the number in the wheel it drives, as their circumferences, *i. e.* as their radii. This combination is represented in (Fig. 64.)

To estimate the advantage to be gained by such a combination, let $P, P', P'' \dots P_n$ be the forces applied at the several wheels; $w, w', w'' \dots w_n$, the weights which they would equilibrate at their respective pinions: then the radii of the wheels and pinions, being denoted by $R, R', R'' \dots R_n$, and $r, r', r'', \dots r_n$. There are the following equations,

$$\frac{P}{w}=\frac{r}{R},\ \frac{P'}{w'}=\frac{r'}{R'},\ \frac{P''}{w''}=\frac{r''}{R''}\dots\frac{P_n}{R_n}=\frac{r_n}{R_n}.$$

Multiplying these equations together, and remarking that, in the combination, the force at the first pinion is that applied to the second wheel; the force at the second pinion is that applied to the third wheel, &c., or $w=P', w'=P'', \dots w_{n-1}=P_n$. there is

$$\frac{P}{w_n}=\frac{r.r'.r''\dots r_n}{R.R'.R''\dots R_n}.$$

or, since the force acting at the last pinion is the weight, it is

$$\frac{P}{W}=\frac{r.r'.r''\dots r_n}{R.R'.R''\dots R_n}.$$

This theorem may be expressed by means of the revolutions performed in a given time. For the teeth which

work in each other, being necessarily set at equal distances, the radius of a pinion is to that of the wheel it drives, as the number of teeth in the former to that of the teeth in the latter, *i. e.* as the number of revolutions performed by the wheel to that performed in the same time by the pinion. Thus, if the pinion has six leaves, and the wheel driven by it sixty teeth; the pinion must perform ten revolutions, whilst the wheel makes only one. And universally, the number of revolutions performed in a given time by the pinion and the wheel it drives, are inversely as the number of their teeth, *i. e.* inversely as their radii. Wherefore putting ν, ν', ν'', &c. for the number of revolutions performed by the first, second, third wheel, &c. in the same time, we shall have

$$\frac{r}{R'}=\frac{\nu'}{\nu}. \qquad \frac{r'}{R''}=\frac{\nu''}{\nu'}. \qquad \text{and } \frac{r_{n-1}}{R_n}=\frac{\nu_n}{\nu_{n-1}}.$$

Making these substitutions in equation (a), it becomes

$$\frac{P}{W}=\frac{r_n \cdot \nu_n}{R\nu}.$$

shewing that for equilibrium, the ratio of the power to the weight is had, by multiplying the radius of the last axle and that of the first wheel, each by the number of revolutions performed by it in a given time. So that if $R = 10. r_n.$ and $\nu = 40. \nu_n.$ we should have

$$W = 400. P.$$

and a man working such an engine with a force equal to 56 pounds, would sustain a weight of ten tons.

25. The teeth of the pinions and wheels afford the means not only of communicating motion from one to the other, but also of changing the direction of that motion; as may be seen by figures 65 and 66. In the former of which, the pinion drives a crown wheel in a plane perpendicular to its own plane; and in the latter, the same thing is done by bevelled work.

In the construction of bevelled work, the particulars chiefly to be attended to, are the inclination of the axes, and the tapering form of the teeth. The angle contained by the axes is equal to that which measures the change to be made in the direction of the motion; as may be seen by (Fig. 67.), where the pinion N drives the wheel M. The angle formed by the axes is BAD: and the four-sided figure BADC being right angled at B and D, the angle BAD is equal to ECB, which measures the change of direction.

The manner in which the teeth should taper in bevelled work, will be understood by conceiving two cones to roll on each other, whilst turning round their respective axes. These cones are exhibited in (Fig. 68.) Whilst the surfaces are smooth, the motion cannot be communicated from one to the other, without considerable pressure; and this would be attended with a violent strain on the axle. To prevent this inconvenience, and consequent waste of force, the surfaces are fluted as in (Fig. 69.) and it is evident, that the flutings should converge to the common vertex. Moreover, as the more delicate parts of the flutings, near the angles of the cones, are of little use, those parts may be dispensed with; and then, nothing will remain but the bevelled wheels BEFC, CFGD.

THE PULLY.

26. If a cord is used simply as the means of acting at a distance from the point to which the force is to be applied, the service it performs is often of considerable value; but as the cord, by its tension, acts with equal energy in opposite directions, it is plain that in this use of it, the power to be applied must be equal and opposite to the resistance to be encountered; and, therefore, that no change is made either in its quantity or direction. It is otherwise, when one of those forces being applied to strain the cord in the

direction of its length, the other is applied transversely, to inflect it. Let the cord ACB (Fig. 70.) be drawn by two forces, T, T′, acting at its extreme points A and B; and let it be inflected by the force F, acting at any intermediate point, C, in the direction CE. Then, if any line, DC, in the direction of F, is taken to represent this force, and if this is made the diagonal of a parallelogram HK, whose sides are in the directions of the lines CA, CB, the forces acting in those directions shall be represented by CH, CK or HD, and there will be the proportions expressed by the following equations:

$$\frac{\text{F}}{\text{CD}} = \frac{\text{T}}{\text{CH}} = \frac{\text{T}'}{\text{HD}}. \qquad \text{or,} \quad \frac{\text{F}}{\sin.(\theta + \theta')} = \frac{\text{T}}{\sin.\theta'} = \frac{\text{T}'}{\sin.\theta}.$$

θ, θ' being the angles made by the directions of the forces T and T′, with that of F.

If A and B are fixed points, to which the cord is attached by its extremities, the reactions of these points will take the places of the forces T and T′; and the same analogy will express the relation of the inflecting force F, to the strains on those points. And in either case, the forces T and T′ are the tensions of the two parts of the cord CA, CB.

27. But if the inflecting force F, instead of acting at the same invariable point of the cord, is at liberty to change its point of application; as when it is applied to a running knot, or to a ring, through which the cord is passed, then the two parts of the string communicating freely, their tensions must be equal, *i. e.* $\text{T} = \text{T}'$, and therefore, $\theta = \theta'$. Making these substitutions in the former theorem, there is

$$\text{F} : \text{T} :: \sin. 2\,\theta : \sin.\theta :: 2\cos\theta : 1. \ \text{ or}$$
$$\text{F} = 2\,\text{T}.\cos\theta.$$

If $\theta = 90°$. or ACB is one right line, there shall be $\cos.\theta = 0$, and therefore F, infinitely less than T, for equilibrium. Hence it appears, that if an inextensible cord lies in a right line between its extreme points, a transverse force,

however small, shall overcome any force, however great, by which it is strained longitudinally: and, therefore, that by the most inconsiderable transverse force, there may be produced on the fastenings, a strain which is only limited by the strength of the cord itself. But if the cord suffers extension, cos. θ. is different from cypher, and therefore, F bears some assignable ratio to T. As the angle at C diminishes, cos. θ. increases; and therefore the ratio of F to T increases. When that angle becomes 120°, or $\theta = 60°$, $2 \cos. \theta = 1$, and $\therefore$ F $=$ T. and when the angle at C vanishes, *i. e.* when the two parts of the cord are parallel, $\cos. \theta = 1$, and $\therefore$ F $= 2$ T.

Wherefore, if one end of a cord is fixed, and a weight or resistance is applied at the other end, and the power transversely at some intermediate point, there will be a mechanical advantage, so long as the angle contained by the two parts of the cord exceeds 120°; and this advantage will be greater, as the angle is increased; being infinite, for an angle of 180°, *i. e.* when the two parts of the cord lie in directum. But if the angle made by the two parts of the cord is less than 120°, the mechanical advantage is to be gained by applying the power as a tending, and the resistance as an inflecting force. And as the angle further diminishes, the ratio of the power to the resistance diminishes, becoming that of 1 : 2 when the angle vanishes, *i. e.* when the two parts of the cord are parallel.

28. The loss of force which would be occasioned by the rigidity of the rope, if bent at a sharp angle, is avoided by passing it over or under a grooved wheel. And to lessen the friction, the wheel is made to turn on an axle; or the axle on the points of support. Thus, the friction is transferred from the rope and circumference of the wheel to the axle, where its leverage is so much less.

Such is the pully, consisting of a grooved wheel or sheave, turning on an axle fixed in a block. The rope passing over or under the sheave, two of the three equilibrating forces

are applied at its extremities; and the third at the block which carries the sheave.

The use of this instrument is merely to lessen the effects of rigidity and friction; so that if these were not to be considered, the rope might be supposed to pass round a pin, or through a ring. Wherefore, the theorem stating the relation between the power and resistance already delivered, may be immediately applied to this instrument. The power always acts at one extremity of the rope; and as the resistance or weight may be applied either to the other extremity of the rope, or to the block in which the sheave revolves, the instrument is divided into two kinds, applicable to different purposes. These two kinds are called the fixed and the moveable pully.

29. The fixed pully is represented by (Fig. 71.) where ABD is the wheel or sheave turning on an axle at C, and in the block CE. The rope is MABN; to one end of which is applied the power P, and to the other, the resistance W. The block is attached to a point, either absolutely fixed, or to be so considered.

From Art. 26. it appears, that for equilibrium, P must be equal to W; and, therefore, that no mechanical advantage is gained. The only use of the fixed pully is, to change the direction of the power; the force P, acting in the direction AM, being thereby made to equilibrate the force W, acting in the line BN. The change made in the direction of the force is measured by the angle GFM. And if the two parts of the rope, AM, BN, were parallel, a power acting vertically downwards would draw a weight vertically upwards, *i. e.* the change of direction would be measured by an angle of 180°.

29. If the resistance or weight is applied to the block, as in (Fig. 72.) the pully ascends or descends with the weight it supports; and is, therefore, called the moveable pully: and for equilibrium, we have

$$\text{P} = \frac{\text{W}}{2\cos.\theta}.$$

This relation between P and W may be expressed by means of the circle ADB. For if radii are drawn to the points A and B, where the rope parts from the sheave; the lines CA, CB, AB, are perpendicular to the lines AM, BN, CF, the directions of the three equilibrating forces: and, therefore, the triangle CAB is similar to a triangle, whose three sides are in those directions: wherefore,

$$\text{P} : \text{W} :: \text{CA} : \text{AB}.$$

i. e. the power shall be to the resistance, as the radius of the wheel to the chord of the arch embraced by the rope; so that putting c for this chord, and r for the radius of the sheave, it will be

$$\text{P} : \text{W} :: r : c.$$

If the two parts of the rope are parallel, we shall have

$$c = 2r.$$

and therefore,

$$\text{P} = \frac{\text{W}}{2}.$$

Accordingly, the moveable pully is used with most advantage when the parts of the rope are parallel: the energy of the power being then doubled at the working point.

30. By combining the fixed and moveable pullies, the twofold advantage is gained, of reducing the power by which the weight is to be equilibrated; and of changing the direction in which it is to be exerted. This is represented in (Fig. 73.) where the rope, attached to a fixed point, is passed under a moveable pully, and then over a fixed pully. The power, applied to the end which hangs from the fixed pully, shall sustain a weight of twice its magnitude applied at the block which contains the moveable pully. And if the first end of the rope, instead of being secured by a fixed point, is passed over a second fixed pully, and then attached to the block which carries the weight, as in (Fig. 74.) the

power shall sustain a weight of thrice its magnitude. For the rope being at perfect liberty to move over or under the pullies, must be equally strained in every part. One of those strains is measured by the power, and if the ropes are vertical, the remainder by the weight.

31. Several moveable pullies may be combined in the same block, and as many fixed pullies in another block. In these combinations there is one rope which passes alternately over the fixed, and under the moveable pullies. The power is applied to one end of this rope, whilst the other end is fixed; or else, attached to either block. Such are the combinations represented in (Figures 75 and 76.)

The power is to the weight as unity to the number of strains by which the weight is supported: and this number is, in one case, twice the number of moveable pullies; and in the other case, greater by unity. Wherefore putting n for the number of moveable pullies, it will be

$$\text{P} : \text{W} :: 1 : 2n. \qquad \text{or } \text{P} : \text{W} :: 1 : 2n+1.$$

according as the end of the rope is attached to a fixed point, or to the block which carries the weight.

32. Single pullies or systems of pullies, such as have been described, may become the constituent parts of other systems; and hence arises considerable variety. Thus, to a system, such as that described in the last article, a moveable pully may be added, having a distinct rope, one end of which is attached to the block containing the system of moveable pullies. The other end may be attached to a fixed point, as in (Fig. 77.) Or it may be carried over another fixed pully, and then connected with the block of the pully which it carries, as in (Fig. 78.) The weight is appended from the block of this moveable pully; and it shall be twice as great in the one case, and in the other thrice as great, as the power would sustain, without this addition to the system. For the appended weight it equal to the sum of the strains of the parts of the last rope; and that sustained

by the block above it, is equivalent only to one of those strains.

If to each moveable pully there is a distinct rope, one end of which is attached to the block of the next superior pully, and the other end to a fixed point, the block of the lowest pully carrying the weight, as in (Fig. 79.); the strain on each succeeding rope, beginning with that to which the power is applied, is twice as great as the strain of the preceding rope, and the weight is equivalent to twice the strain of the last rope. Wherefore, n denoting the number of moveable pullies, the relation between the power and weight shall be

$$\text{P} : \text{W} :: 1 : 2^n.$$

If the end of each of these ropes, instead of being attached to a fixed point, is carried over a fixed pully, and then attached to the block which it carries, as in (Fig. 81.) the ratio of the power and weight will be given by the analogy.

$$\text{P} : \text{W} :: 1 : 3^n.$$

The energy is yet greater, in either of these systems when inverted: the ends of the ropes, or the pullies which before were fixed, being now attached to the weight, as in (Figures 81 and 82.) Thus, in (Fig. 81.) the strain of the rope attached to the weight at B, is P; that of the rope at C, is 2 P; that of the rope at D, is 4 P; and that of the rope E, is 8 P. Wherefore, the weight W, being equal to the sum of these strains, is P, multiplied into the sum of the terms of a geometrical series whose first term is unity, and whose ratio is 2, the number of terms being one more than the number of moveable pullies. Consequently the equilibrating weight is expressed by the equation $\text{W} = (2^{n+1} - 1)\,\text{P}$. In the arrangement represented in (Fig. 82.) the strain at B is 2 P; that at C is 2. 3 P; that at D, is 2. 3. 3 P; and that at E is 2. 3. 3. 3 P. Wherefore, the weight is P, multiplied by the

sum of a geometrical series whose first term is 2, and whose ratio is 3; the number of terms being, as before, one more than the number of falling pullies. Accordingly, the equilibrating weight is expressed by the equation $W = (3^{n+1} - 1)P$.

33. These several arrangements may be variously combined; and from what has been delivered, the weight which the power will carry in each, is easily computed. This may be exemplified in the combinations known by the name of the Spanish Burtons, represented in (Fig. 83.) In the first of these, a rope, fixed at one end, passes under a moveable pully which carries the weight; then over another pully; and to this rope the power is applied. But the second pully, instead of being fixed, is carried by a distinct rope, which, passing over a fixed pully, is attached by its other end to the lower block. Here the strain of the second rope is double of that to which the power is applied, or 2 P: and this strain is applied in supporting the weight. Moreover, the rope to which the power is applied, being doubled about the lower pully, exerts the same force of 2 P in sustaining the weight. Wherefore,

$$W = 4P.$$

In the second Burton, a rope applied at one end to the weight, passes over a fixed pully. The other end carries a moveable pully having a distinct rope, which, being applied to it at one end, passes under a second moveable pully, which is attached to the weight, then over the former, and the power is applied to the end of this rope. The strain of the former rope is evidently thrice that of the latter, or 3 P; and the second carries a portion of the weight equal to 2 P. Wherefore,

$$W = 5P.$$

34. In this account of the pully, it has been supposed that the several ropes are strained, in directions parallel to that of the weight or resistance which they support. If any

part of a rope encompassing a moveable pully, is inclined to the direction of the rope it carries, the energy with which it acts against the resistance is to be estimated, by multiplying its strain into the cosine of that angle of inclination. For by resolving the oblique strain into two, one of them parallel, and the other perpendicular to the direction of the strain on the pully, the former is the energy with which it acts in sustaining that pully; and this is the strain of the rope, multiplied into the cosine of the angle of inclination. The latter is the strain multiplied into the sine of the same angle; and this is counteracted by the opposite strain of the other part of the same rope.

If one end of the rope that carries the pully is fixed, the strain on the pully is supported by two strains of the rope, and these are inclined at equal angles. In this case it was shewn, that the strain of the rope was to that supported by the pully, as the radius of the sheave to the chord of the arch embraced by the rope, Art. 29. This affords a ready method of computing the ratio of the power and weight, in a system, consisting entirely of such pullies. Thus, in a system of fixed and moveable pullies, with a single cord fixed at one end, putting r, r', r'', &c. for the radii of the moveable pullies, and c, c', c'', &c. for the chords of the arches embraced by the ropes, we shall have for any obliquity of the directions,

$$\frac{\text{P}}{\text{W}} = \frac{r.r'.r''.\ \&\text{c.}}{c.c'.c''.\ \&\text{c.}}.$$

35. When the power and weight act in the same direction, it is evident that the pressure on the fixed point is equal to W + P; and if the power acts in the direction opposite to that of the weight, that this pressure is W − P. If there are several such fixed points, this strain is distributed among them, as in the system represented in (Fig. 79.) where the points B, C, D, E sustain the several pressures, P, 2 P, 4 P, 8 P, and the point A the pressure 2 P.

THE INCLINED PLANE.

36. If a material point is urged obliquely against a plane, it may be maintained in a state of rest, by the application of another force; provided that the resultant of the two forces is directed perpendicularly against the plane. For such a resultant would be completely counteracted by the resistance or reaction of the plane. Thus, let the plane be CD, (Fig. 84.) against which, a material point at o, is urged by the oblique force Fo; and let oy be perpendicular to the plane, at the side opposite to that on which is placed the material point: the equilibrium shall be established by any force, which, compounded with Fo, gives a resultant in the direction of oy.

To find the force or forces which will satisfy this condition, it is to be observed, that the direction of the sustaining force must be in the plane in which are Fo, oy. Wherefore, if the line AB be the intersection of this plane with the given plane CD, it may be taken for this latter plane. If then FE be drawn perpendicular to AB, *i. e.* parallel to oy, the only condition is, that one of the components being represented in quantity and direction by Fo, the direction of the resultant must be parallel to FE, and it is plain that this condition will be satisfied by a force represented in quantity and direction by a line drawn from the point o, to any point such as M, in the perpendicular FE, produced indefinitely beyond the line AB.

Hence it appears, that the sustaining force, when directly opposed to Fo, is equal to this force: that it is least, when directed in the plane itself, being then represented by oE: that at equal angles above and below the plane, the sustaining forces are equal; and that no force, of whatever magnitude, acting in the direction oy, perpendicular

to the plane, would suffice to sustain the material point against it.

The force Fo, is resolvable into the forces FE perpendicular to the plane, and Eo in that plane; and in like manner, the force oM into the forces EM and oE. Of these, the forces Eo, oE, being equal and opposite, destroy each other, and FE, EM, being in the same line coalesce into one force FM, which is the pressure on the plane. This pressure is the sum or difference of the forces FE, EM.

To express these results algebraically, let P designate the force Fo, and S the sustaining force oM, then the sides of the triangle FoM, being as the sines of the opposite angles, the forces P, S, are reciprocally proportional to the sines made by their directions with the perpendicular to the plane, *i. e.* to the cosines of the angles made by those lines with the plane. Wherefore, a, a', denoting those angles, there is

$$\text{P}. \cos. a. = \text{S}. \cos. a'. \quad (1)$$

which is the condition of equilibrium.

Also, denoting the resultant or pressure on the plane by R, there is

$$\text{R}. \cos. a' = \text{P}. \sin. (a + a'). \quad (2)$$

The force P remaining unaltered in magnitude or direction, it appears from equation (1), that S. cos. a' is constant, or that the sustaining force raises inversely as the cosine of the angle, which its direction makes with the plane. It is therefore least when cos. $a' = 1$. or $a' = o$. The sustaining force is, therefore, most economised when directed in the plane itself, its magnitude being then S $=$ P. cos. a.

37. Such, in general, is the theory relative to a material point supported against a plane. But there is a particular case which deserves to be considered distinctly. It is, that wherein the plane is inclined to the horizon, and the force with which the point is urged is its weight: such is denominated the inclined plane.

The whole theory of the inclined plane flows at once from what has been thus generally established: for in this case, the force P is the weight of the body whose direction is vertical; and therefore, the angle formed by it with the plane, is the complement of that made by the plane with the horizon. Denoting this latter angle, or the elevation of the plane by ε; and the weight of the body by W, there will be $P = W$. $a = 90 - \varepsilon$. Also, putting γ for the angle made by the direction of the sustaining force and plane, there will be $\cos. a = \sin.\varepsilon$. $\cos. a' = \cos.\gamma$. $\sin.(a + a') = \sin.(90^\circ - \varepsilon + \gamma) = \cos.(\gamma - \varepsilon)$. Wherefore, substituting these values in (1) and (2), there will be

$$S = W \frac{\sin.\varepsilon}{\cos.\gamma}. \qquad R = W. \frac{\cos.(\gamma - \varepsilon)}{\cos.\gamma}.$$

giving the values of the sustaining force and pressure.

When the sustaining force acts in the direction of the plane, there is $\gamma = o$. and therefore,

$$S = W. \sin.\varepsilon. \qquad R = W. \cos.\varepsilon.$$

Or putting l, h, b for the length, height, and base of the plane, there shall be

$$S = W. \frac{h}{l}. \qquad R = W. \frac{b}{l}.$$

When the direction of the sustaining force is parallel to the base of the plane, or $\gamma = \varepsilon$. there shall be

$$S = W. \tan.\varepsilon. \qquad R = \frac{W}{\cos.\varepsilon}.$$

or,

$$S = W. \frac{h}{b}. \qquad R = W. \frac{l}{b}.$$

On account of the importance of this subject it may not be amiss to establish this theory by a separate investigation. Let AC be the inclined plane (Fig. 85.), BC its height, and AB its base. Then if from the right angle B, the line Bo is

let fall perpendicular to the plane, and from the point C, the line Cm is drawn parallel to the direction of the sustaining force, meeting the line Bo at m, the three sides of the triangle CBm, being parallel to the directions of the three forces, shall be proportional to them in magnitude. Wherefore S, R, W denoting, as before, the sustaining force, the pressure and the weight, there shall be

$$\text{S} : \text{W} :: m\text{C} : \text{CB}.$$

$$\text{R} : \text{W} :: \text{B}m : \text{CB}.$$

When the sustaining force is parallel to the plane, mC coincides with oC.

But

$$o\text{C} : \text{CB} :: \text{CB} : \text{CA} :: h : l.$$

and

$$o\text{B} : \text{CB} :: \text{BA} : \text{CA}. \ b : l.$$

Therefore,

$$\text{S} = \text{W}. \frac{h}{l}. \qquad \text{R} = \text{W}. \frac{b}{l}.$$

When the direction of the sustaining force is parallel to the horizon, mC is perpendicular to CB, and

$$m\text{C} : \text{CB} :: \text{CB} : \text{BA} :: h : b.$$

$$m\text{B} : \text{CB} :: \text{CA} : \text{BA} :: l : b.$$

and

$$\text{S} = \text{W}. \frac{h}{b}. \qquad \text{R} = \text{W}. \frac{l}{b}.$$

These results are readily put into an analytical form, for the sides of the triangle BmC being as the sines of the opposite angles, if ε and γ denote as before the elevation of the plane and the angle made by it with the direction of the sustaining force, and ζ is put for the angle made by this same line of direction with the base, there will be

$$m\text{C} : \text{CB} :: \sin.m\text{BC} : \sin.\text{B}m\text{C} :: \sin.\varepsilon : \cos.\gamma.$$

$$m\text{B} : \text{CB} :: \sin.m\text{CB} : \sin.\text{B}m\text{C} :: \cos.\zeta : \cos.\gamma.$$

$$\therefore \text{s} = \text{w}.\frac{\sin.\varepsilon}{\cos.\gamma}. \qquad \text{r} = \text{w}.\frac{\cos.\zeta}{\cos.\gamma}.$$

which results agree with those already given.

38. If instead of a material point, it is a body of any definite magnitude, whose weight is to be supported on the inclined plane, certain other conditions are to be satisfied respecting the position of the body, and the direction of the force to be applied.

With respect to the position of the body, it is to be observed, that its weight and the reaction of the plane are to be equilibrated by the sustaining force. But the forces of resistance being directed in lines perpendicular to the inclined plane at the points of contact, the direction of their resultant is perpendicular to that plane, at some point of the surface of the polygon, formed by connecting the points of contact. Wherefore, if the plane passing through the centre of gravity of the body, perpendicular to the inclined and horizontal planes, *i. e.* perpendicular to the intersection of those planes, does not pass through the surface of contact, the forces to be equilibrated are not in the same plane; they have then no single resultant, (Sect. III. Art. 3.) and consequently, they cannot be equilibrated by a single force. Wherefore, to render the body capable of being sustained by the application of a single force, the plane passing through its centre of gravity, and perpendicular to the intersection of the inclined and horizontal planes, must pass through the surface of contact.

Such is the condition relative to the position of the body. If this condition is not satisfied, it will be requisite to apply two forces, which must be such, that the resultant of the weight and two sustaining forces shall be directed towards the plane, in a perpendicular at some point of the surface of contact.

The condition relative to the position of the body, being fulfilled, the body may be sustained by the application of a single force directed in the plane passing through the centre

of gravity, and perpendicular to the intersection of the inclined and horizontal planes. But the force applied for this purpose is restricted by a condition relative to its point of application.

To see the nature of this restriction, let CD be the inclined plane, (Fig. 86.); *gv* the vertical line passing through the centre of gravity; *abc* the surface of contact, intersected by the plane of the forces in the line *ab*. Then, if from *a* and *b*, the extremities of this line, perpendiculars are raised to the plane CD, they shall meet the vertical *gv*, as at *d* and *e*. And as the perpendiculars to the plane at the several points of the line *ab* must necessarily pass through the points of the line *de*, it follows, that the direction of the sustaining force must pass through some point of the line *de*.

39. If two planes of equal heights are placed back to back, as in (Fig. 87.), and if the weights W, W′ are laid on them, connected by a cord passing over a pully at the highest point, so that the parts of the cord shall be parallel to the planes; it will be requisite for equilibrium that the cord should be drawn equally by the two weights, *i. e.* that the parts of those weights, which act in the directions of the planes, should be equal. Wherefore, putting L, L′ for the lengths of the planes, and H for the common height, the weights must be such as to satisfy the equation $W.\frac{H}{L} = W'.\frac{H}{L'}$. or $\frac{W}{L} = \frac{W'}{L'}$. *i. e.* the weights which equilibrate in this manner, must be to each other, as the lengths of the planes on which they are placed.

40. If the body is to be supported by two planes which meet at an angle, its weight is to be equilibrated by their reactions: accordingly, the directions of those three forces must meet at a point. And from hence are derived the conditions relative to the planes themselves, and to the position of the body to be supported.

The plane of the forces, being perpendicular to each of the inclined planes, must be perpendicular to their intersection. The same plane is also vertical; inasmuch as it contains the vertical line passing through the centre of gravity of the body. Therefore, the intersection of the inclined planes must be perpendicular to a vertical plane, *i.e.* it must be horizontal. This condition, relative to the positions of the planes, being satisfied, the body is to be placed so that the plane passing through its centre of gravity, and perpendicular to the intersection of the inclined planes, shall pass through each of the surfaces of contact. If the body is not so placed, it will not compose itself until this condition is satisfied; and then, the pressure on each of the planes shall be to the weight, as the sine of the elevation of the alternate plane, to the sine of the angle made by the planes. And these, which are the total pressures on the planes, are to be distributed on the points of contact, as already stated.

THE WEDGE.

41. The wedge is a triangular prism, *i.e.* a solid, bounded by two equal and parallel triangular planes, and three rectangles contained between their parallel sides. Two of these inclined faces are to be introduced between the bodies to be separated; and the angle, made by these faces, is called the angle of the wedge. The rectangular surface, opposite to this angle, is the back of the wedge, and to this the power is applied.

Of all mechanical instruments, this may appear to be the most simple: yet in accounting for its operation some variety will be found, arising from the diversity in the nature of the forces, both of those to be encountered, and of those by which they are to be surmounted. In each of the instruments already treated of, the force to be encountered was ever applied in the same manner. But the resistances to be

overcome by the wedge, may be forces applied immediately to its faces, as in separating two distinct bodies; or to its edge, as in cutting; or they may proceed from parts situated beyond the edge, as in splitting or cleaving. And according to the way in which these forces are applied, the several parts of the instrument will be more or less effective. Also, the power applied may be either simple pressure, or that arising from percussion. But though the force brought into action by percussion is, in general, that which is most effectively applied to the wedge, yet as the treatment of percussion belongs to Dynamics, the force employed as a power must, for the present, be regarded as a pressure or weight.

As the wedge is forced between the resistances, these may be supposed to slide along its faces; and if this is done without attrition, the faces will transmit only those forces which act in lines perpendicular to those surfaces. Wherefore, the resistances, when applied to the faces of the wedge, can be equilibrated by the power, only when they act in directions perpendicular to those faces. The same thing is to be understood of the faces of a cleft, into which the wedge has fully entered: these surfaces, being regarded as without friction, can receive an impression from the wedge, *i. e.* from the power, only in lines perpendicular. Wherefore, in all cases, the power is to be resolved in directions perpendicular to the surfaces through which the forces are to be transmitted; and the resistances are supposed to act in the opposite directions.

The resistances being applied to the faces of the wedge, let this be represented by its triangular end ABC, and let DF, perpendicular to the back of the wedge, represent the power, (Fig. 88.) Then if this line is made the diagonal of a parallelogram GE, whose sides, DG, DE, are perpendicular to the faces AC, BC, it is evident that the force DF will be in equilibrio with the forces of resistance represented by GD, and ED or FG. But the lines DF, FG, GD, being perpendicular to AB,

BC, AC, the triangle DFG is similar to the triangle ABC: wherefore, the power is to the sum of the forces which it will equilibrate, as AB to AC + BC.

If the triangular end of the wedge is isosceles, as represented in (Fig. 89.) the power is to the sum of the resistances, as half the back of the wedge to one of the sides, *i. e.* as the sine of half the angle of the wedge to radius.

Hence it follows, that the more acute the angle of the wedge, the greater the resistance which it will overcome by the application of a given power; and that the efficacy of the instrument, estimated in this way, is inversely as the sine of the semiangle.

42. In cleaving, the motion is along the faces of the cleft, and therefore, the power is to be resolved in directions perpendicular to those surfaces, and the resistances act in the opposite directions. Let the angle of the cleft be ADB, (Fig. 90.); and putting B for AB, the base of the triangular end of the wedge; and L for BD, the depth of the cleft measured from the points A or B, along its faces; the energy with which the power acts against the resistance, is $P.\frac{L}{B}$. But as the resistance yields, not in a right, but in a curved line, whose centre of curvature is at the angle of the cleft, it is evident, that to estimate the efficiency of the instrument in this use of it, we must take into account the leverage by which the forces act. Now, the resistance to be overcome is the cohesive force of the fibres, which are more and more extended from the angle D, to the point of fracture. The point D is, therefore, the fulcrum: and putting l for the distance from this to the point where the strains of the fibres may be supposed to be concentred, $R.l$ shall be the moment of the resistance: and $\frac{P.L^2}{B}$, is the moment of the force to which it is opposed. Wherefore, for equilibrium,

$$\frac{P.L^2}{B} = R.l. \qquad \text{or, } P = \frac{R.l.B}{L^2}.$$

which shows in what way the efficacy of the wedge depends on the constitution of the substance to be divided, as also, on the size of the instrument. Being at present concerned in estimating the power of the wedge, we shall suppose the material to be given. In this case, R is given: also, the angle of the cleft, or $\frac{B}{L}$, is constant; and, therefore, l is likewise constant. Wherefore, the power to be applied varies inversely as L, *i. e.* inversely as B, the back of the wedge; and the efficiency of the instrument varies directly as that quantity.

Such is the advantage gained by the size of the wedge when applied to a substance that yields in this manner. When once the faces of the cleft part from those of the wedge, the accuracy of the angle of the instrument becomes of no value. And the same may be said of the magnitude of that angle after that the wedge has fully entered the cleft. But in cutting, its efficacy mainly depends on this part. The smaller the angle at which the faces are inclined, and the more accurately they are brought to an edge, the less is the number of parts whose cohesive strength is to be encountered. And if it is a soft or flexible substance that is to be divided, the effect is to be ascribed to the angle exclusively. For then, the faces of the incision, by reason of their flexibility, are incapable of transmitting the forces applied to them by the power, to the part of the substance beyond the edge, which remains to be divided. Those faces being then incapable of acting as levers, the effect is altogether independent of the depth of the wedge. The advantage arising from the sharpness of the angle, in dividing a flexible substance, will further appear in the next section.

THE SCREW.

43. The screw is an instrument consisting of two parts: one of them is a solid cylinder; the other, a hollow cylinder of the same diameter. The former of these is encompassed with a salient thread, proceeding round it in the form of a spiral: and in the latter is a spiral groove, in which it may be lodged. The solid cylinder is more especially called the screw, and the hollow cylinder the nut. It is evident that the screw cannot advance within the nut, or the nut upon the screw, but in the direction of the spires or threads, *i. e.* by a motion compounded of two motions; one of them circular, the other progressive. When one of these parts is moved on the other, in this manner, it will press against any obstacle by which its further progress is impeded. To ascertain the force thus exerted by the moveable part, in relation to the power applied to move it, the form of the spiral thread must be more particularly considered.

Let abc be a right angled triangle, whose base bc is equal to the circumference of the base of the cylinder, (Fig. 91.) Then, if the line ab is applied to one of the sides of the cylinder, and the triangle wrapped round it, the point c shall reach the point b, and the hypothenuse ac shall become one circumvolution of such a spiral. If the same thing is done with a right angled triangle, whose base is $2cb$, and whose altitude is $2ab$, its hypothenuse shall mark out two circumvolutions of the same spiral; and if the base and altitude are $3cb$, $3ab$, three circumvolutions shall be marked on the surface of the cylinder, and so forth.

The spiral formed by the hypothenuse of a right angled triangle, when its surface is thus wrapped round a cylinder, is by geometricians called a helix. The base of the triangle being taken equal to the circumference of the base of the cylinder, its altitude is the interval between the spires mea-

sured on the side of the cylinder, *i. e.* in a direction parallel to its axis.

This may be represented in a manner somewhat different, thus: let a rectangular parallelogram be described, whose altitude is the side of the cylinder, and whose base is equal to the circumference of its circular base, as in (Fig. 92.) Let this rectangle be divided by the lines $m'n'$, $m''n''$, &c. parallel to the base, and at the equal distances nn', $n'n''$, &c. and let the transverse lines mn', $m'n''$, &c. be drawn. Then, if the side of this rectangle is applied to the side of the cylinder and the surface wrapped round it, the line mn' shall become the first spire; and the point n' falling on the point m', the line $m'n''$ shall become the second spire, and the point n'' falling on the point m'', the line $m''n'''$ shall become the third spire, and so forth: the several lines mn', $m'n''$, $m''n'''$, $m'''n''''$, &c. becoming one continued spiral, proceeding from one end of the cylinder to the other.

44. The thread of the screw is then an inclined plane, carried round the surface of a cylinder: for which reason, the ratio of the power to the resistance in equilibrio, is furnished by the theory of the inclined plane. The direction of the resisting force is plainly that of the axis of the cylinder or height of the plane. With respect to the relative motion of the parts, the moving power may be applied indifferently to the screw to move it within the nut, or to the nut in the opposite direction to advance it on the screw: and the resistances equilibrated by the same power, applied in these two ways, shall be equal, but in opposite directions. Wherefore, to assimilate this to the case of an inclined plane, on which a weight is to be supported, the power may be supposed to be applied to the nut, to carry it up a vertical screw, against a resistance acting downwards. This power is applied in a direction parallel to cb, the base of the inclined plane, (Fig. 91.) And if it were applied immediately at the surface of the cylinder, the power would be to the weight as

the height of the plane to its base. Putting h for ab, and c for the circumference of the base of the cylinder,

$$\text{P} : \text{W} :: h : c. \qquad \text{or, } \text{P} = \frac{h}{c}.\ \text{W}.$$

But the power is usually applied to the arm of a lever inserted in the moveable part, as in (Fig. 93.) The fulcrum being at the centre of the circular motion, *i. e.* at the axis of the cylinder, if R is put for the length of the arm to which the power is applied, the energy of that power at the surface of the cylinder shall be $\text{P}.\frac{\text{R}}{r}$. giving

$$\text{P}.\frac{\text{R}}{r} = \frac{h}{c}\ \text{W}.$$

or,

$$\text{P} = h.\frac{r}{\text{R}.c}.\ \text{W}$$

$$= \frac{h}{\text{C}}.\ \text{W}.$$

C being the circumference of circle whose radius = R. *i. e.* the power is to the resistance which it equilibrates by means of the screw, as the interval between the spires, to the circumference of the circle described by the power. It will be observed, that this proportion is altogether independent of r, the radius of the cylinder.

The screw, therefore, as it is commonly used, combines the advantages of the lever and inclined plane. In reasoning on that part of the instrument which works on the latter principle, it has been supposed that the power and resistance are both applied at a single point of the plane; or, which is the same thing, that the screw and nut touch in a single point. In fact, they touch throughout the entire extent of the thread of the nut, or, at least, in many separate points. But however great the number of the points of contact, the relation between P and W is the same. For the whole of the resistance W may be distributed into the parts m, m', m'', &c. en-

countered at those points: and if the power P is, in like manner, distributed into parts n, n', n'', &c. such that

$$n = \frac{h}{c}.\ m.$$

$$n' = \frac{h}{c}.\ m'.$$

$$n'' = \frac{h}{c}.\ m''.\ \&c.$$

the equilibrium shall subsist at each point; and therefore, throughout the whole extent of the plane. And by adding, we have

$$n + n' + n'' + \&c. = \frac{h}{c}.\ (m + m' + m'' + \&c.)$$

that is,

$$P = \frac{h}{c} W.$$

45. The mechanical advantage afforded by this instrument is increased, either by augmenting the length of the lever handle, or by diminishing the interval between the spires. But the augmentation of the length of the lever handle is an increase of bulk, which it is desirable to avoid. And to the diminution of the interval between the spires, there is a limit, arising from the necessity of leaving sufficient strength in the parts to endure the enormous pressure, and consequent friction, without breaking. The device of Mr. Hunter, published in the Philosophical Transactions of the year 1781, seems well fitted to give any power to the instrument, without either of these disadvantages. In this instrument, represented by (Fig. 94.) the screw which turns within the nut, is not solid as usual, but is itself the nut of a somewhat finer screw, on which it turns.

To shew the power of this contrivance, let h be the interval between the threads of the exterior, and h' that between the threads of the interior screw. Now by one revo-

lution of the exterior screw it would be carried forward through the space h, and the interior screw, if firmly attached to it, would be carried through the same space. But the revolution of the interior screw is prevented by a cross bar by which it is perforated, and which bears against the frame work, wherefore that screw will perform one revolution relatively to the exterior screw, and in the opposite direction, and therefore shall move upwards through the space h'. The space actually described by the interior screw is the difference of these, or $h - h'$. Thus, when $h = \frac{1}{10}$ of an inch and $h' = \frac{1}{11}$. the progressive motion is $\left(\frac{1}{10} - \frac{1}{11}\right)$ of an inch $= \frac{1}{110}$th of an inch.

It appears then, that the instrument, in this example, combines the mechanical advantage of a screw, the interval between whose threads is but the $\frac{1}{110}$th of an inch, with the strength of one whose threads are ten times as gross: and this, without increase of bulk, or any disadvantage, except what may arise from increase of friction.

It would seem that this contrivance might be beneficially adopted in the construction of micrometer screws, and a degree of accuracy thereby given to our observations, altogether unattainable by the most delicate instruments now in use. Thus, if the exterior screw has 50, and the interior screw 51 spires to an inch, we should obtain the advantage of a screw of the ordinary construction, having 2550 spires to an inch. In general, the number of spires to an inch, in the equivalent screw of the common construction, is the product of the numbers of the spires to an inch in the two members divided by their difference.

46. There is another modification of this instrument, called the endless screw: so named, because by turning, it never comes to an end. It is evident that this advantage

can be obtained, only when the motion of one of the parts is circular. The screw, in this instrument, consists of a few spires; and these work in the teeth of a wheel, which, therefore, serves in place of a nut; as represented by (Fig. 95.) The screw has a circular, but not a progressive motion. By the action of the spire engaged, the tooth of the wheel would proceed forward: but the wheel is limited to a circular motion round its axis; and by this motion the tooth is soon disengaged from the spire, and then its place is supplied by another tooth, which is acted on as the former; and this tooth is succeeded by a third in the same manner; and so on perpetually.

The mechanical effect of this combination is thus estimated.

Let C, as before, be the circumference of the circle described by the winch handle, R its radius; h, the interval between the spires. Also let R′ be the radius of the toothed wheel, and r that of its axle. Then P′ being the force equilibrated by the screw alone, there is

$$\mathrm{P} = \frac{h}{\mathrm{C}}\,\mathrm{P}'.$$

But the force P′ is immediately applied at the circumference of the toothed wheel, and equilibrates the weight applied at its axle. Wherefore

$$\mathrm{P}' = \frac{r}{\mathrm{R}'}.\,\mathrm{W}.$$

Multiplying these equations, there is

$$\mathrm{P} = \frac{h}{\mathrm{C}}.\,\frac{r}{\mathrm{R}'}.\,\mathrm{W}.$$

Now h, the interval between the spires of the screw, is also the interval between the teeth of the wheel: wherefore if n denotes the number of teeth, and C′ the circumference of the wheel, there is $nh = \mathrm{C}'$, and $h = \frac{\mathrm{C}'}{n}$. This value of h

being substituted in the last equation, it becomes

$$\mathrm{P} = \frac{1}{n} \cdot \frac{\mathrm{C}'}{\mathrm{C}} \cdot \frac{r}{\mathrm{R}'} \cdot \mathrm{W}.$$

or since

$$\frac{\mathrm{C}'}{\mathrm{C}} = \frac{\mathrm{R}'}{\mathrm{R}},$$

it is

$$\mathrm{P} = \frac{1}{n} \cdot \frac{r}{\mathrm{R}} \cdot \mathrm{W}.$$

i. e. the power is to the weight, as the radius of the axle which carries the weight, to the radius of the circle described by the winch handle, multiplied by the number of teeth in the wheel.

SECTION VIII.

THE FUNICULAR POLYGON AND CATENARY.

1. If a system of forces is in equilibrio when applied to a body of variable form, it is evident that the equilibrium shall not be violated should the body become rigid. Hence it follows, that the conditions to be satisfied for the equilibration of forces applied to a rigid body, should also be satisfied in the case of a body of variable form. But it is equally evident that, in this latter case, these conditions are not sufficient, *i. e.* that other conditions are to be fulfilled in order that the body may become capable of opposing to each other the forces of the system. These conditions relate both to the nature and the form of the body. With respect to the former, it is plainly requisite, that the body should be capable of transmitting the applied force from one point to another, and, therefore, that the parts between the points of application should be either inextensible or incompressible, at least to a degree sufficient to resist the action of those forces.

If the body is simply inextensible, it shall be capable only of transmitting divellent forces, *i. e.* forces which tend to increase the distances between the points of application: and if it is simply incompressible, it shall serve only for the transmission of compressing forces. A flexible but inextensible cord affords a familiar example of a body of the former kind. Let o, o', o'', o''', o'''', be the points of such a cord (Fig. 96.) to which the forces p, p', p'', p''', p'''', are applied; these forces must be divellent: therefore, the forces p, p'''', applied at the extreme points, must act in the directions $o' o$,

$o''' o''''$, and the forces p', p'', p''', applied at the intermediate points, must be directed to the exterior side of the polygon. If the lines oo', $o'o''$, $o''o'''$, $o'''o''''$, represent a system of rigid bars barely apposited, they will serve only for the transmission of compressing forces, *i. e.* of forces directed towards the interior of the polygon, or contrary to the former. But if the bars are connected by hinges, the system may serve indifferently for the transmission of distending or compressing forces.

2. When two forces are applied at the ends of a cord, and others transversely at so many intermediate points, as described in the preceding article, the figure assumed by the cord is that denominated the funicular polygon. Let the cord be that exhibited in the figure; and o, o', o'', o''', o'''', the points to which are applied the forces p, p', p'', p''', p'''', acting in the same or in different planes. To investigate the conditions to be satisfied in order that these forces should equilibrate, we shall first suppose the equilibrium to subsist. The conditions which follow from that supposition will be necessary.

Because of the flexibility of the cord, the forces can act against each other only in the lines oo', $o'o''$, $o''o'''$, $o'''o''''$. Let these forces be represented by the lines ko, $o's$, $o''s'$, $o'''s''$, $r'o''''$. and let all between the first and last be resolved, each in the directions of the branches of the cord contiguous to its point of application, as os into $o'm$, $o'k$; $o''s'$ into $o''n$, $o''m'$; $o'''s''$ into $o'''n'$, $o'''r$. and as the forces applied at the ends of the entire cord are necessarily in the directions of the first and last branches, the whole system of forces p, p', p'', p''', p'''', shall be replaced by another system of forces directed along the sides of the polygon: and if the equilibrium exists among the forces of the former, it must continue to subsist among those of the latter system.

But among these it cannot subsist unless each part of the

cord is drawn equally in opposite directions. This condition gives

$$ok + o'k' = 0. \quad o'm + o''m' = 0. \quad o''n + o'''n' = 0. \quad o'''r + o''''r' = 0.$$

Because these forces are the tensions of the parts, they may be denoted by t, t', t'', &c. and the system shall be $t - t + t' - t' + t'' - t'' + t''' - t'''$. whose sum is cypher. These tensions may be different for the different branches of the cord, the tension of any branch being the resultant of all the forces acting at either side of that branch: for the force p or t, acting at o, and in the direction of the line oo', may be supposed to be immediately applied at the point o', where, if compounded with the force p' or $t' - t$, the resultant shall be $+t'$, and this resultant of the forces p, p', being in the direction of $o'o''$, may be supposed to be immediately applied at o'', where, if compounded with the force p'', or $t'' - t'$, the resultant, which is that of the three forces p, p', p'', shall be t''. In like manner, beginning at the other end of the entire cord, the force p'''' or $-t'''$ may be supposed to be immediately applied at the point o''', where, if compounded with the force p''' or $t''' - t''$, the resultant shall be $-t''$. Thus it appears that the equal forces by which the branch $o''o'''$ is strained in opposite directions, are the resultants of the groups of forces acting at each side of that branch; or that the resultants of the forces p, &c. taken from either end of the entire cord to any angle of the polygon, is in the direction of the portion next ensuing, and that it constitutes its tension.

Further: because the equilibrium is supposed to exist in the entire cord, it must exist among the forces acting at each angular point. But these consist of one of the forces p, applied immediately at that point, and of the tensions of the adjacent branches of the cord; and those two tensions being the resultants of the groups preceding and following, may be replaced by those forces; whence it follows, that the whole

system of forces p, p', p'', p''', p'''', must equilibrate at the same point. Thus the forces acting at the point o'' are t', $-t''$ and p'', and if these three are in equilibrio at that point, the equilibrium shall continue when the forces t' and $-t''$ are replaced by their components; but t' is the resultant of the forces p, p'; and $-t''$ of the forces p''', p''''. Hence it follows, that any system of forces in equilibrio when acting against each other by means of a funicular polygon, would also equilibrate about any material point to which they may be transferred in parallel directions.

The same thing appears by the tensions into which those forces have been resolved; these tensions being equal and opposite in pairs.

It now appears that a system of forces cannot equilibrate by means of a funicular polygon, unless the following conditions are satisfied. 1st, The forces must be such as would equilibrate about a material point to which they may be transferred in parallel directions. 2d, Each branch of the cord must be parallel to the direction of the resultant of the forces acting at either side of that branch. Further, it is evident, that when these conditions are satisfied by the forces applied, and the form of the polygon, the equilibrium must necessarily subsist. For the first branch being taken in the direction of the force p, that force may be supposed to have been immediately applied at the point o'; and the second branch being taken in the direction of the resultant of the forces p and p' acting at o', this resultant may be supposed to be immediately applied at the point o'', and so forth. In the same manner it may be shewn, that the resultant of all the remaining forces acts at the same point o''; and the same may be shown of any other angle of the polygon.

It appears, then, that when this form is given to the polygon, the whole system of forces acts at each angle; and that if these forces are such as would equilibrate about a material

point, the equilibrium shall necessarily subsist at each angle of the polygon.

It is now proved, that for equilibrium among forces acting against each other by means of a funicular polygon, it is requisite and sufficient that the forces should be such as would equilibrate about a material point to which they may be transferred in parallel directions, and that each branch of the polygon should be taken parallel to the direction of the resultant of the group of forces from either end of the entire cord, to the nearer extremity of that branch inclusively.

3. Seeing that when the applied forces, taken from either end of the entire cord to any of its angles inclusively, are transferred in parallel directions to a material point, and there compounded, the resultant is equal to the tension and parallel to the direction of the succeeding branch; it follows that the tension and direction of any branch are given by the same equations which determine the magnitude and direction of the resultant of a system of forces acting on a material point. Thus let there be three rectangular axes meeting at a point, and let a, a', a''.........a_n. be the angles made by the several branches of the cord with the axis of x; b, b', b''...........b_n. those which they make with the axis of y; and c, c', c''...........c_n. those made with the axis of z. Likewise let a, α, α'...........a_n; b, β, β'...........b_n; c, γ, γ'...........c_n, be the angles made by the directions of the forces p, p', p''........p_{n+1}, with the same axes. Then as the tension of any member, as t'', is the resultant of the forces p, p', p''. there shall be

$$\left.\begin{aligned} t''.\cos.a'' &= p.\cos.a + p'.\cos.\alpha + p''.\cos.\alpha'. \\ t''.\cos.b'' &= p.\cos.b + p'.\cos.\beta + p''.\cos.\beta'. \\ t''.\cos.c'' &= p.\cos.c + p'.\cos.\gamma + p''.\cos.\gamma'. \end{aligned}\right\} \quad (1)$$

Or, putting X$''$, Y$''$, Z$''$, for the second members, *i. e.* for

the sums of the forces p, p', p'', reduced to the directions of the axes of x, y, z, the equations shall be

$$\left.\begin{array}{l} t''.\cos.a'' = \text{X}''. \\ t''.\cos.b'' = \text{Y}''. \\ t''.\cos.c'' = \text{Z}''. \end{array}\right\} \qquad (2)$$

Whence by squaring and adding, there are

$$t'' = \sqrt{\text{X}''^2 + \text{Y}''^2 + \text{Z}''^2}. \qquad (3)$$

and

$$\cos.a'' = \frac{\text{X}''}{\sqrt{\text{X}''^2+\text{Y}''^2+\text{Z}''^2}}. \quad \cos.\ b'' = \frac{\text{Y}''}{\sqrt{\text{X}''^2+\text{Y}''^2+\text{Z}''^2}}.$$

$$\cos.c'' = \frac{\text{Z}''}{\sqrt{\text{X}''^2+\text{Y}''^2+\text{Z}''^2}} \qquad (4)$$

These equations applied to the last branch give the value and direction of t_n, which are already known; yet these equations are not therefore useless, for replacing t_n with its known value, $-p_{n+1}$, in equations (1), they become

$$p.\cos.a + p'.\cos.\alpha' + p''.\cos\,\alpha'' \ldots\ldots\ldots + p_{n+1}.\cos.a_n = 0.$$
$$p.\cos.b + p'.\cos.\beta' + p''.\cos.\beta'' \ldots\ldots\ldots + p_{n+1}.\cos.b_n = 0.$$
$$p.\cos.c + p'.\cos.\gamma'' + p'.\cos.\gamma'' \ldots\ldots\ldots + p_{n+1}.\cos.c_n = 0.$$

Which express the condition to be satisfied in order that the equilibrium may be possible.

If the cord is attached to a fixed point at one end, the equations give the tension and direction of the last branch; and as in this case, the number of equations is only equal to that of the unknown quantities, the problem is not restricted by any condition, and the equilibrium is therefore always possible.

If the cord is attached to fixed points at both ends, the equilibrium is possible *a fortiori.* In this case it is not solved by these equations, which are fewer than the number of unknown quantities, of which there are eight, viz. p and

p_{n+1}. And the angles a, b, c, α_n, b_n, c_n. But the problem is not therefore indeterminate, for beside the three equations (1) there are the two following, viz.

$$\cos.^2 a + \cos.^2 b + \cos.^2 c = 1. \quad \cos.^2 a_n + \cos.^2 b_n + \cos.^2 c_n = 1.$$

with three others arising from this, that the fixed points are given in position. For taking one of those points for the origin of the coordinates, those of the other fixed point may be expressed as functions of the quantities which enter into the other equations, and of the lengths of the several branches of the cord, which last are given. The expressions to which this computation would lead are exceedingly complicated; and as their value would not reward the labour of the analyst, it seems sufficient thus briefly to point out the method by which the figure of the polygon may be completely ascertained.

4. The extreme points being fixed, if the forces applied at the intermediate points are parallel, the sides of the polygon, and therefore, also, the directions of the forces must all be in the same plane. For the plane of any two contiguous sides contains the direction of the force applied at the angle made by those sides, and the plain of any two successive parallel forces contains the intermediate side of the polygon. Thus, the plane $oo'p'$ is the same with the plane $p'o'o''$ (Fig. 99.) And the forces p', p'' being parallel, the plane $p'o'o''$ is the same with the plane $p''o''o'$. The same reasoning may be continued through the whole system of lines.

If the forces are appended weights, their resultant acts in the vertical passing through their common centre of gravity. And as the reactions of the fixed points are in equilibrio with this resultant, it follows, that the directions of the two extreme portions of the cord shall meet at some point of the vertical passing through the common centre of gravity of the weights. And similarly, the tensions of any

two parts of the cord, regarded as divellent forces, being in equilibrio with the weights applied at the intermediate points, those parts, if produced, shall meet at some point of the vertical passing through their common centre of gravity. Thus, oo', $o''''o'''$ produced, shall meet at the same point of the vertical passing through the common centre of gravity of the weights p', p'', p''', and oo', $o'''o''$, produced, shall meet at some point of the vertical passing through the common centre of gravity of the weights p', p''. Wherefore, the theorem which expresses the relations of the three forces engaged at any one angle of the polygon may, in this case, be extended to the strains of any two parts, however remote, and the sum of the weights applied at the intermediate angular points, *i. e.*

$$t : t''' : p' + p'' + p'''$$
$$:: \sin. c' : \sin. a : \sin. (a + c').$$

The case of a cord loaded with weights being that which offers itself most to our notice in this theory, it may not be amiss to shew how the tensions and weights may be represented by a simple construction.

From any point, o, let there be drawn the lines oo', oo'', oo''', &c. (Fig. 100.) respectively parallel to the sides of the polygon, and meeting the vertical line in the points o', o'', o''', &c. then the three forces t, p', $-t'$, which are in equilibrio about the point o', (Fig. 99.) shall be represented by the lines oo', $o'o''$, and $o''o$. Likewise, the three forces t', p'', and $-t''$, in equilibrio about the point o'', shall be represented by the lines oo'', $o''o'''$, and $o'''o$: and so of the three forces which are in equilibrio at any other angular point.

Moreover, $+t'$ and $-t'$, which are of the same magnitude, being represented by the same line oo'' or $o''o$, it follows, that the sides of the triangles $oo'o''$, $oo''o'''$, shall serve, not only to exhibit the relations of the three forces acting at

one and the same point, o' or o'', but also to compare the forces acting at one of those points with those which act at the other: and the same thing being observed of all the triangles, it follows that there is this proportionality.

$$t : p' : t' : p'' : t'' : p''' : t''', \text{ \&c.}$$
$$:: oo' : o'o'' : oo'' : o''o''' : oo''' : o'''o'''' : oo'''', \text{ \&c.}$$

Wherefore, if the directions of the sides of the polygon are known, together with any one of the appended weights, all the other forces, both weights and tensions, are absolutely determined.

Drawing the horizontal line oh, it appears that the tensions of the several parts of the cord are to one another as the secants of the angles of their inclinations to the horizon: and that the weights appended at the several angular points are as the differences of the tangents of those angles. Also, that the horizontal strain, which in each part of the cord acts equally in opposite directions, is represented in magnitude by oh, and is therefore the same throughout the entire system.

5. Hitherto the cord has been considered, only, as the means of conveying the actions of the forces, to points remote from those to which they are immediately applied; and for this reason it was regarded as being, itself, without weight. But the cord and its minutest parts having weight, it is plain, that if it is fastened at its two ends, those points not being in the same vertical, it must be deflected from a right line by the weights of its several parts; which, in this position of the cord, are so many forces applied transversely to its length. The number of these forces being infinite, that of the sides of the polygon must also be infinite, *i. e.* the figure assumed by such a cord is a curve.

This curve is known by the name of the catenary; and its properties follow immediately from those of the funicular

polygon loaded with weights, merely by supposing the number of those weights to be infinite.

As it will be sufficient to consider one branch of this curve, its lowest point may be supposed to be fixed; and, therefore, the other branch may be cut away. This being premised, let *ov* be one branch, terminating in the lowest point, *v*, (Fig. 101.) in which, taking any point, *m*, and drawing the tangent *ms*, and the horizontal line *mh*, let these lines meet the vertical passing through the point *v*, at *s* and *h*; then putting *e* for the angle *hms*; W for the weight of the cord between *m* and *v*; T for the tension at the point *m*; and A for the tension at *v*, the relative magnitudes of T, A, W shall be represented by *ms*, *mh*, *hs*. Wherefore,

$$\mathrm{W} = \mathrm{A}.\frac{hs}{mh} = \mathrm{A}.\tan. e. \qquad (a)$$

$$\mathrm{T} = \mathrm{A}.\frac{ms}{mh} = \frac{\mathrm{A}}{\cos. e} \qquad (b)$$

Showing that the weight between any point *m*, and the lowest point, varies as the tangent of the inclination to the horizon at the former point; and that the tension varies inversely as the cosine, *i. e.* directly as the secant of the same angle.

If the cord is uniform, the curve is called the homogeneous catenary, in which $\mathrm{W} = ks$; *s* denoting the length measured to the lowest point; and *k* the weight in a unit of length. Wherefore,

$$\mathrm{A}. \tan. e = ks. \qquad (c)$$

From which it appears, that the two branches into which the homogeneous catenary is divided by the lowest point, are precisely similar.

Taking the point *v* for the origin, and the horizontal line through this point, *vr*, for the axis of the abscissæ; and the vertical *vh* for the axis of the ordinates, there will be for any point of the curve, as *m*,

$$vq = x, \qquad mq = y, \qquad \text{and } \tan. e = \frac{dy}{dx}.$$

Whereby equation c becomes

$$\text{A}.dy = k.s.dx. \tag{d}$$

And as $ds = \sqrt{(dy^2 + dx^2)}$, any of the three quantities, y, x, s, may be eliminated, and an equation obtained between the remaining two.

To render the expressions more simple, let $\text{A}' = \frac{\text{A}}{k}$; whereby equation ($d$) becomes

$$\text{A}'dy = s.dx.$$

which, being differenced, gives

$$\text{A}'d^2y = \sqrt{(dy^2 + dx^2)}.dx.$$

And multiplying both sides by $\frac{dy}{\sqrt{(dy^2+dx^2)}}$, it will be

$$\frac{\text{A}'.dy.d^2y}{\sqrt{(dy^2+dx^2)}} = dy.dx.$$

that is,

$$\text{A}'.d.\sqrt{(dy^2 + dx^2)} = dy.dx.$$

Wherefore, by integration,

$$\text{A}'.\sqrt{(dy^2 + dx^2)} = ydx + \text{C}dx.$$

For the lowest point, v, there will be

$$y = 0. \qquad dy = 0.$$

which gives

$$\text{C} = \text{A}'.$$

and therefore,

$$\text{A}'\sqrt{(dy^2 + dx^2)} = (\text{A}' + y)\,dx. \tag{e}$$

for the differential equation of the homogeneous catenary. This being integrated, gives

$$x = \text{A}'.\log.\left(\frac{\text{A}' + y + \sqrt{(2\text{A}'y + y^2)}}{\text{A}'}\right).$$

From which it appears, that the homogeneous catenary is a transcendental curve.

To find equation between y and s, let dx be eliminated between the equations $A'dy = s.dx$, and $ds = \sqrt{(dy^2+dx^2)}$. and we shall have

$$\frac{A'^2.dy^2}{s^2} = ds^2 - dy^2.$$

or

$$(A'^2 + s^2)\,dy^2 = s^2.\,ds^2.$$

and

$$\frac{s.\,ds}{\sqrt{(A'^2 + s^2)}} = dy.$$

And by integration,

$$\sqrt{(A'^2 + s^2)} = y + C.$$

But for $y = 0$. there will be $s = 0$. Wherefore,

$$C = A'; \qquad \text{and } \sqrt{(A'^2 + s^2)} = y + A'.$$

so that

$$s^2 = y\,(2A' + y).$$

which shews that the homogeneous catenary is a rectifiable curve; its length from any point to the lowest, being a mean proportional between y and $2A' + y$. Wherein y is the ordinate of the highest point of the portion to be rectified.

6. If any of the forces, as p''', is applied to a ring, or loop running on the cord, its direction must bisect the angle made by the adjacent branches $o''o'''$, $o'''o''''$. For if the equilibrium subsists, it shall not be violated by fixing the points o'', o'''': and then the ring is confined to the surface of an ellipsoid whose foci are those points, and whose major axis is equal to $o''o''' + o'''o''''$. On this surface the ring cannot rest unless the direction of the force is normal: *i. e.* unless it bisects the angle made by lines drawn from the point of application to the foci. These angles being equal, the

tensions of the branches $o''o'''$, $o'''o''''$, also must be equal: and if all the forces p are applied at rings, the tension must be the same throughout the entire cord. This indeed is evident, inasmuch as the whole is then one cord; and from this consideration, what has been proved relative to the directions of the transverse forces would have immediately followed: for the tensions of $o''o'''$ and $o'''o''''$ being equal, they must make equal angles with the direction of the force p''' by which those tensions are equilibrated. This is the condition of a cord strained round a polygon. The transverse forces are the reactions of the angular points, the reaction at any point being equal to the tension multiplied by twice the cosine of half the angle. Thus, putting ω for the angle, there is $p = 2\,t.\cos.\frac{\omega}{2}$.

If the sides of the polygon are indefinitely small, and their number indefinitely great, the polygon becomes a curve. The tension of the cord is in every part the same, and therefore equal to the force by which it is strained at each end; but the transverse force acting at any point, *i. e.* the reaction of that point of the curve, and, therefore, the pressure upon it is incomparably smaller than the tension. This appears from the preceding equation, for considering the curve as a polygon, with an indefinite number of sides, the angle $\frac{\omega}{2}$. becomes a right angle whose cosine is cypher. Wherefore, in the case of a curve, the transverse force, to be comparable with the tension, must be taken, not for a mathematical point, but for some definite portion of the curve.

To obtain the pressure on an element of the curve, which shall be supposed to be of single curvature, let N be the pressure made by a unit of length. This may vary from one element of the cord to another, but may be considered as unvaried in the same element ds. Now, referring this ele-

ment to the axes of x and y, as the tension at any point is the resultant of all the forces applied to the cord from the beginning up to that point, it follows that the tension, reduced to any one direction, is the sum of all the applied forces reduced to the same direction. Wherefore, α being the angle formed by the tangent at the first point of ds, any element of the cord, with the axis of x, and a being the angle made by the direction of the tangent at the beginning of the curve, with the same axis there is $-t.\cos.\alpha = t\cos.a + \int \text{N}.\sin.\alpha.ds.$ and differencing, $t.d\alpha = \text{N}.ds.$ Now, ω being the angle made between the perpendiculars to the tangents at ds, and the beginning of the curve, there is $d\alpha = d\omega$. Wherefore, by last equation, $t.d\omega = \text{N}ds.$ But $d\omega = \frac{ds}{\text{R}}.$ wherein R is the radius of curvature. Wherefore,

$$\text{N} = \frac{t}{\text{R}}.$$

which shows that the pressure on the curve is every where inversely proportional to the radius of curvature.

If the cord is strained round a cylinder with a circular base, there is $\text{R} =$ constant, and $\text{N} =$ constant. In this case the equation $\text{N}ds = t.d\omega$, by integration, gives $\text{N}s = t\omega + \text{C}.$ and as at the beginning of the arch there is $s = o$, $\omega = o$. there is also $\text{C} = o$. wherefore the equation is

$$\text{N}s = t\omega.$$

which shows that the pressure on any arc s, is the product of the tension into the angle of incurvation; viz. into the angle made by two perpendiculars to the arc, raised at its two extreme points; and that the pressure made on the cylinder by each circumvolution of the cord, is $2\pi t$. viz. the tension multiplied by the number expressing the ratio of the circumference of a circle to its radius.

SECTION IX.

OF THE EQUILIBRIUM OF ROOFS, ARCHES, AND DOMES.

1. To support a beam by two forces applied at its ends, the forces, if not vertical, should be directed to some point of the vertical passing through the centre of gravity of the beam: and further, the two forces and weight should be proportional to the sines of the angles contained by their directions; each force being as the sine of the angle contained by the directions of the other two forces. Therefore, if the beam is given, together with the direction of one of the sustaining forces, the whole is given: for if the given line of direction is produced to meet the vertical passing through the centre of gravity of the beam, the line, drawn from the other end to this intersection, gives the direction of the other sustaining force; and each is to the weight, as the sine of the angle made by the other sustaining force and vertical, to the sine of the angle contained by the directions of the sustaining forces.

The two following examples are given, as of much importance in practice.

2. A beam rests with one end against a wall: the pressures on the points of support are required.

Let the beam be AB, (Fig. 102.) resting against the wall at A; and let *o* be its centre of gravity. The wall being supposed to be perfectly smooth, its reaction is according to the perpendicular to its surface, at the point A. Let this perpendicular be AG, meeting the vertical passing through *o*, at G. Now if the end B were to rest on a horizontal plane, and if this plane were also perfectly smooth, its reaction

would be vertical; and the beam could not possibly be supported by these reactions. To support the beam, the force applied at B must be in the direction BG: and then, if the horizontal line, Bn, is drawn, meeting the vertical passing through o, at n, the three equilibrating forces shall be as the sides of the triangle BGn, parallel to their directions. And if the force BG is resolved into two, one of them vertical and the other horizontal, these will be nG, Bn, the former equal and opposite to the weight, and the latter to the reaction of the wall.

Putting F, W, T for the force BG, the weight and the horizontal thrust, we shall have

$$\mathrm{F} = \mathrm{W} \cdot \frac{\mathrm{BG}}{\mathrm{AD}}. \qquad \mathrm{T} = \mathrm{W} \cdot \frac{\mathrm{B}n}{\mathrm{AD}}.$$

Moreover, making AB : oB : : a : 1. we shall have

$$\mathrm{B}n = \frac{\mathrm{BD}}{a} \qquad \text{and } \mathrm{T} = \mathrm{W} \cdot \frac{\mathrm{BD}}{a . \mathrm{AD}} = \frac{\mathrm{W}}{a . \tan . \mathrm{ABD}}.$$

Whereby it appears, that in a given beam, the horizontal thrust varies inversely as the tangent of its inclination to the horizon; and accordingly, that for the horizontal position, the thrust is infinite.

If the span BD is given, the thrust shall vary as the weight divided by the vertical height of the point A, above the point B; *i. e.* if the weight of the beam is as its length, the thrust shall be inversely as the sine of the inclination to the horizon.

3. If the beam rests on a prop at the upper end, as on the edge of a wall, the reaction of the prop shall be perpendicular to the beam. Wherefore, drawing the line AG perpendicular to AE, meeting at G the vertical on, passing through o, the centre of gravity, (Fig. 103.) the direction of the support at B must be BG. And the line Bm being drawn parallel to AG, the three equilibrating forces shall be as the sides of the triangle GBm, parallel to their direc-

tions; *i. e.* putting P for the reaction of the prop, we shall have

$$P = W.\frac{mB}{Gm}. \qquad F = W.\frac{BG}{Gm}. \qquad \text{and } T = W.\frac{Bn}{Gm}.$$

Making $AB = a.OB.$ we shall have

$$Gm = a.om.$$

and

$$P = W.\frac{mB}{a.om} = W.\frac{\cos. ABD}{a}.$$

$$T = W.\frac{Bn}{a.om} = \frac{W}{a(\tan. + \cot.) ABD}.$$

and T shall be greatest when (tan. + cot.) ABD is least; *i. e.* when ABD $= 45°$. Its value being then, $\frac{W}{2a}$. from which it diminishes to cypher, as the beam approaches either to the vertical or the horizontal position.

If the beam is bisected by its centre of gravity, $a = 2$. and at the elevation of 45°, $T = \frac{W}{4}$.

If the span, $Bn = \frac{BD}{a}$, is given; and if W varies as the length of the beam, the thrust shall vary as $\frac{BO}{om}$; *i. e.* as the sine of the horizontal elevation of the beam.

The case, considered in the last article, is that of a rafter abutting against another, the top or ridge being at liberty to descend in the vertical. The present article relates to a rafter loosely supported on a prop at the upper end, and at liberty to slide from it, in the direction of its length.

4. If the funicular polygon loaded with weights is inverted, it is evident that the weights, decomposed according to the directions of the sides, shall become thrusts instead of tensions, *i. e.* compressing in place of divellent forces; and that these thrusts are to be communicated from one point to another, by incompressible bars, instead of inextensible cords.

The weights applied at the angles, when resolved in the directions of the bars, yield components depending on the angles of inclination, as in the funicular polygon: and when the equilibrium is established, the thrusts in each separate bar are equal and opposite. Wherefore, putting thrusts for tensions, the theory of the equilibration of apposited bars is the same as that of the funicular polygon loaded with weights; and established by the same reasoning. The weights, it is true, are never placed at the angles exclusively: but the vertical pressure on any bar or beam, including the weight of the beam itself, may be referred to the extreme points; *i. e.* to the angles of the polygon. For the portion of the weight bearing on either extremity, is to the entire weight, as the distance of the centre of gravity from the other extremity to the entire length of the beam. The vertical pressure made on the same point, by the beam which constitutes the other side of the angle, is found in the same manner; and the sum of these is the pressure on the intermediate angular point. In a roof covered with materials of uniform thickness and density, the weight on each angular point is half the sum of the weights on the containing sides.

Thus, let o, o', o'', o''', o'''' be the polygonal roof (Fig. 104.); p, p', p'', p''', p'''' the weights incumbent on the angular points, or suspended from those points. Then, the weights at each angle being resolved in the directions of the rafters which meet at that point; it is requisite and sufficient for equilibrium, that the thrusts in each bar should be equal: t, t', t'', t''' being those thrusts, we have

$$p'.\frac{\sin.a}{\sin.(a+a')}=t'=p''.\frac{\sin.b'}{\sin.(b+b')}.$$

or,

$$p'.\frac{\sin.a}{\sin.a.\cos.a'+\sin.a'.\cos.a}=p''.\frac{\sin.b'}{\sin.b'.\cos.b+\sin.b.\cos.b'}.$$

that is,

$$\frac{p'}{\cos.a' + \sin.a'.\cot.a} = \frac{p''}{\cos.b + \sin.b.\cot.b'}.$$

Or dividing the denominators by $\sin.a'$, $\sin.b$, which are equal, it will be

$$\frac{p'}{\cot.a + \cot.a'} = \frac{p''}{\cot.b + \cot.b'}.$$

and so of the rest.

It is to be observed of the angles of the polygon, except that at the highest point, that the parts into which they are divided by the vertical lines, are, one of them acute and the other obtuse; and, therefore, that the co-tangents of those parts are affected with opposite signs. The theorem, therefore, agrees, as it should, with that for the funicular polygon; where we had

$$\frac{p'}{\tan.e - \tan.e'} = \frac{p''}{\tan.e' - \tan.e''}, \text{ \&c.}$$

$\frac{p'}{\cot.a + \cot.a'}$ $(= t'. \sin.a')$ is the horizontal thrust: whereby it appears, that the horizontal thrusts are equal throughout the system.

The number of equations thus supplied, is one less than the number of angles in the polygon, *i. e.* n being the number of rafters, it is $n - 2$. Putting m, m', m'', &c. for the lengths of the rafters; s for the span; and h for the height or pitch, we have also the equations

$$s = m.\sin.a + m'.\sin.b + \text{\&c.} \qquad h = m.\cos.a + m'.\cos.b + \text{\&c.}$$

the rafters, concerned in the second of these equations, being those on one side of the highest point. If the polygon is symmetrical, the number of distinct equations is less than that of the unknown quantities. For example, if it is proposed to construct an equilibrated roof of four equal rafters, m, (Fig. 105.) the structure being symmetrical about

the vertical passing through the highest point o'', it will be sufficient to consider one half; and for this we have, from the conditions of equilibrium,

$$\cot.a + \cot.a' = 2\cot.b.$$

but

$$\cot.a' = -\cot.b.$$

wherefore,

$$\cot.a = 3\cot.b.$$

Moreover,

$$h = m(\cos.a + \cos.b). \qquad s = 2m(\sin.a + \sin.b).$$

The quantities here concerned are m, s, h, and the angles a and b; and as there are but three equations, it is requisite that two of these five quantities should be given: one of them being a line.

5. In constructing an arch of masonry, a piece of frame work, called a centre, is first erected, whose circumference corresponds to the figure of the intended arch. On this are placed truncated wedges, called voussoirs, beginning at the piers or abutments, and finishing at the top or crown. On the voussoirs are laid other materials suitable to the particular purpose; whether to support a road way, an aqueduct, or a building. The centering is then removed, and the whole abandoned to the mutual pressures of its parts. The voussoir at the crown is called the key-stone; the interior curve the intrados; and the exterior the extrados.

In treating of the equilibration of such a structure, the faces of the voussoirs are supposed to be perfectly smooth, and therefore incapable of transmitting lateral pressure, except in directions perpendicular to the joints.

The tendency to descend, in each voussoir, by its weight and that of the load it carries, is to be equilibrated by the lateral thrusts of the voussoirs on each side: or, which is the same thing, those weights, resolved in directions perpendicular to the joints, should yield components equal and opposite in pairs, *i. e.* the weight of each voussoir, including that of its load, should be resolvable into forces, equal and

opposite to the pressures against its faces. Accordingly, the theory of the arch may be derived from that of the wedge, or of the catenary. Because of the importance of the present subject, it is proposed to consider it without reference to those theories.

Let the arch be that represented in (Fig. 106.) A, B, C, D, &c. the voussoirs; ab, $a'b'$, &c. the joints; and cd, $c'd'$, &c. the vertical lines passing through the centres of gravity of the several voussoirs, and the weights with which they are loaded.

The pressure made by the key stone, against the face of the contiguous voussoir, is the resultant of the pressures made by all the points of its face ab; and this resultant may be applied at any point of that face. If m is that point, let mo, perpendicular to ab, be continued to meet the line $c'd'$ at o'; and let $o'm'$, perpendicular to $a'b'$, meet the line $c''d''$ at o''; and $o''m''$, perpendicular to $a''b''$, meet the line $c'''d'''$ at o'''; and let $o'''m'''$ be perpendicular to $a'''b'''$, &c. The broken line, $oo'o''o'''m'''$, shall be the line of pressure for one side of the arch; and it is similarly made out for the other side. As a voussoir can support a pressure, only when directed perpendicularly against its face, it follows, that if the line of pressure passes above the lower voussoirs, the upper part of the arch must spread over them; the arch descending at the crown, and rising at the haunches: and that if the line of pressure passes below the lower voussoirs, the arch shall fail in the opposite way, *i. e.* by rising at the crown, and descending at the haunches.

The line of pressure may vary through a space, depending on the depth of the voussoirs; and this affords the means of providing against the failure of the structure in this way. Thus let two curve lines be described perpendicular to the joints; one of them passing through the upper, and the other through the lower extremity of the key stone. The condition, relative to the line of pressure, requires that

every joint should be contained, in the whole or in part, between those curves; and this condition is the more easily secured, the greater the depth of the voussoirs.

If this condition is not satisfied, the forces cannot be opposed to each other, and therefore the equilibrium cannot subsist; but this being satisfied, it may be established: and for this it is requisite, that the resolved forces should be equal to those by which they are to be severally supported.

To develope this second condition, let the line CD be vertical, and DD_n, horizontal; and let the lines CD′, CD″, CD‴, &c. be parallel to the joints *ab*, *a′b′*, *a″b″*, &c. Then, in order to consider one-half of the arch, let the voussoir, A, be supposed to be divided into two equal parts, by the line CD. The weight of the half voussoir, *cabd*, is perpendicular to DD′; and the forces into which it is resolved, are perpendicular to CD, CD′. Wherefore, this weight and the forces, into which it resolved, shall be represented by DD′, CD, CD′. In the same way, if the weight of the voussoir B, is represented by D′D″, the forces into which it is resolved shall be CD′, CD″. And conversely, if these are the resolved forces, D′D″ shall be the weight of B; and so of the rest.

This statement is altogether independent of the supposition of equilibrium among the several voussoirs. But for equilibrium, it is requisite that the thrusts should be equal, as well as opposite. Wherefore, if the weight of half the voussoir A, is represented by DD′, and accordingly the forces into which it is resolved by CD, CD′, one of those into which the weight of B is resolved, must be likewise represented by CD′; and, therefore, its weight, and the other of the forces into which it is resolved, by D′D″ and CD″. And the thrust from above, against the joint *a′b′*, being CD″, this must also represent the opposite thrust from below; and accordingly, the weight of C, and its resolved forces shall be D″D‴, CD″, CD‴. Thus the lines of the figure shall serve

to compare, not only the weight of each voussoir with the forces into which it is resolved, but all the forces promiscuously: the weight of the half voussoir A, and those of the rest being represented by DD′, D′D″, D″D‴, &c. and the thrusts at the several joints by the lines CD′, CD″, CD‴, &c. Wherefore, if w', w'', w''', &c. are the weights; and a', a'', a''', the angles made by the joints with the vertical, we shall have

$$\frac{w'}{\tan. a'} = \frac{w''}{\tan. a'' - \tan. a'} = \frac{w'''}{\tan. a''' - \tan. a''} = \frac{w' + w'' + w'''}{\tan. a'''}.$$

i. e. the weights of the several voussoirs are as the differences of the tangents of the angular inclinations of their faces to the vertical; and the weight taken from the crown to any joint, as the tangent of the angle made by that joint with the vertical.

Putting t', t'', t''', &c. for the thrusts perpendicular to the joints, these are as the secants of the same angles, *i. e.*

$$\frac{t'}{\sec. a'} = \frac{t''}{\sec. a''} = \frac{t'''}{\sec. a'''} = \&c.$$

And if the weight of the arch from the crown to any joint is known, the thrust is given by the equation

$$t = \frac{w}{\sin. a}.$$

wherein a and t relate to any joint indifferently; and w denotes the weight of the arch, from the crown to that joint.

The thrust perpendicular to a joint, multiplied by the cosine, or divided by the secant of the inclination of that joint to the vertical, is the horizontal thrust: and as the thrusts perpendicular to the joints vary as those secants, the horizontal thrust is the same throughout the entire system;

and, therefore, equal to the thrust at the crown. Its value is given by the equation

$$h.t = t.\cos.a = w.\frac{\cos.a}{\sin.a} = \frac{w}{\tan.a}.$$

The horizontal thrust is therefore equal to the weight of the arch, taken from the crown to the joint inclined to the vertical in an angle of 45°.

All this is agreeable to what has been found relative to the funicular polygon or catenary; the horizontal elevations of the cord, in its several parts, being the inclinations of the perpendiculars on the cord to the vertical.

6. On these principles the curve of the extrados might be assigned, when that of the intrados is given, together with the depth of the work on the crown; and *vice versâ*. But the application of these principles to practice is somewhat precarious, owing to the general uncertainty of the real conditions of the problem to be solved. The three following are given as extreme cases, of which all others are more or less compounded.

7. Let it be proposed to find the extrados; the voussoirs extending to this line, *i. e.* filling the space between the upper and lower curves.

The joints may be supposed to be every where perpendicular to the curve of the intrados; and the voussoirs themselves indefinitely thin wedges; in which case, their faces shall converge to the centres of curvature, as represented in (Fig. 107.), where o, o', are centres of curvature; the angles at those points being indefinitely small and equal.

Putting w for the weight of the arch, measured from any joint to the crown, as before, and θ for the angle made by the same joint with the vertical, we shall have

$$\frac{d.w}{d\theta} = \text{A}.\frac{d.\tan.\theta}{d\theta} = \text{A}.\sec.^2\theta. \quad (1)$$

Moreover, if r is put for the radius of curvature at any point

I, of the intrados; and l for the distance of the extrados from the centre of curvature, we shall have the sector

$$OIK = \frac{r^2.\,d\theta}{2}, \qquad \text{and the sector } OGH = \frac{l^2.\,d\theta}{2}.$$

and the voussoir IGHK $= \dfrac{(l^2 - r^2)\,d\theta}{2}$. wherefore,

$$l^2 - r^2 = A \sec.^2\theta.$$

omitting the multiplication by 2, as A is a constant quantity as yet to be determined.

At the crown, sec. $\theta = 1$. wherefore, putting l' and r' for the analogous quantities belonging to this point, we have

$$A = l'^2 - r'^2.$$

and therefore,

$$l^2 - r^2 = (l'^2 - r'^2)\sec.^2\theta.$$

giving

$$l^2 = r^2 + (l'^2 - r'^2)\sec.^2\theta. \tag{2}$$

If y denotes the perpendicular from the extremity of the line l, upon the horizontal line drawn through the centre of curvature, we shall have

$$l^2 = y^2 \sec.^2\theta.$$

whereby

$$y^2.\sec.^2\theta = r^2 + (l'^2 - r'^2)\sec.^2\theta.$$

or,

$$y^2 = \frac{r^2}{\sec.^2\theta} + (l'^2 - r'^2).$$

which, for $\theta = 90^\circ$. is

$$y^2 = l'^2 - r'^2.$$

shewing, that when the curve springs at right angles from a horizontal plane, it has an asymptote parallel to that plane, at a distance equal to $\sqrt{l'^2 - r'^2}$. below which it does not descend.

If the curve of the intrados is a circle, $r' = r$. and equa-

tion (2) may receive this form,

$$l^2 = l'^2 + (l'^2 - r^2) \tan.^2\theta.$$

In flat arches, such as are formed of brickwork over the windows of dwelling houses, the joints all converge to a common centre; and therefore, the differences of the tangents are as the breadths of the voussoirs themselves, measured either on the upper or lower edges. Accordingly, the weights of the voussoirs being proportional to their breadths, the vertical heights must be equal, *i. e.* the upper line of the arch must be also right.

The extreme joints, at the angles of the opening, are usually inclined to each other in an angle of 60°; therefore, the distance from the angles to the centre, is equal to the width of the opening; and this, for windows, is generally about five feet. Accordingly, the circle passing through the angles, and perpendicular to the joints, rises above the reveal, by a space equal to 5^{f}.vers.sin.30°, which is somewhat more than 8 inches; and the line of pressure shall not be contained within the voussoirs, unless they are at least equal to this in depth.

8. Let the voussoirs be supposed to be of evanescent depth, *i. e.* let the series of wedges be reduced to a line, which is the curve of the intrados; and let the pressure of the incumbent materials be vertical.

This is the supposition on which the theory of the arch is commonly founded. It differs most widely from the cases which present themselves to the engineer: yet the conclusions, derived from this supposition, may afford him some elements for his calculations, in what relates to the pressures of the loose materials, with which the arch is frequently loaded. For this reason it is here briefly developed.

Let s be the arch measured from any point to the crown; ds shall be its element: and having $d.\theta = \frac{d.s}{r}$. equation (1) may be changed into

$$\frac{d.w}{d.s} = \frac{\text{A}}{r}. \sec.^2\theta. \tag{3}$$

Moreover, taking the span at the abutments for the axis of x, we shall have

$$ds = dx. \sec. \theta.$$

Wherefore,

$$\frac{d.w}{d.x} = \frac{\text{A}}{r}. \sec.^3\theta. \tag{4}$$

If h is put for the height of the work, *i. e.* for the vertical distance between the intrados and extrados, we have

$$d.w = h.dx. \qquad \text{and } h = \frac{\text{A}}{r}. \sec.^3\theta.$$

To find the value of A, we have at the crown, sec. $\theta = 1$. Wherefore, putting h' for the height of the work, and r' for the radius of curvature at this point, we shall have, for the same point,

$$h' = \frac{\text{A}}{r'}. \qquad \text{and A} = h'.r'.$$

Using this value of A, the equation is

$$h = \frac{h'r'}{r}. \sec.^3\theta. \tag{5}$$

If the height of the extrados is required, above the horizontal line on which the span is measured; putting y for the ordinate of the intrados, and Y for that of the extrados for the same abscissa, the equation will be

$$\text{Y} = y + h = y + \frac{h'r'}{r}. \sec.^3\theta. \tag{6}$$

If the intrados is a semicircle, we have

$$r = r', \qquad \text{and } \sec. \theta = \frac{r}{y}.$$

whereby the value of h becomes

$$\frac{h'r^3}{y^3}.$$

and the equation of the extrados,

$$Y = y + \frac{h'.r^3}{y^3}.$$

This curve may descend from the crown to a distance greater or less, depending on the value of h'; but it has a vertical asymptote at the extremity of the horizontal radius, in approaching which, it rises to an indefinite height. The same consequence follows from the supposition made in this article, for every arch which springs from the horizon at right angles.

To find the point where the curve of the extrados cuts its horizontal tangent, we have only to make $Y = r + h'$ in the last equation; which then becomes,

$$r + h' = y + \frac{h'r^3}{y^3}.$$

and this gives the value of h', the depth of work at the crown, for any proposed extent of horizontal road way.

Thus, for $y = \frac{r}{2}$, *i. e.* for an arch of 120°, the equation gives $h' = \frac{r}{14}$.

To find the curve of the extrados, that of the intrados being a semiellipse having the lesser axis vertical.

Let a be the greater, and b the lesser semiaxis. Then,

$$r' = \frac{a^2}{b}.$$

The radius of curvature varies as the cube of the normal, which is

$$y \text{ sec. } \theta. \ i.\ e. \qquad r : \frac{a^2}{b} :: y^3.\text{sec.}^3\theta : b^3.$$

which gives

$$r = \frac{a^2}{b^4}.y^3.\text{sec.}^3\theta.$$

Substituting these values of r' and r, in equation (5,) it is

$$h = \frac{h'.b^3}{y^3}.$$

Were the greater semiaxis vertical, the equation would be

$$h = \frac{h'a^3}{y^3}.$$

The expressions, in both cases, being similar to that already obtained for the circular arch.

9. The depth of the voussoirs being still supposed evanescent, if the curve of the intrados were loaded with a fluid, the perpendicular pressures on the equal elements of the curve would be as the vertical heights of the fluid; and, therefore, the distances of the several points of the intrados below the fluid surface would be, every where, reciprocally proportional to the radii of curvature, (Sect. VIII. Art. 6.) And from this principle the intrados can be determined by two successive integrations.

10. The principle of equilibration in a dome, except in what relates to the horizontal thrusts, is contained in equation (3), viz.

$$\frac{dw}{ds} = \frac{\text{A}}{r}.\text{sec.}^2\theta.$$

To apply this principle, let vABC, (Fig. 108.) be a portion included between two planes intersecting at the axis vo, at an indefinitely small angle; and let this portion be supposed to be insulated, *i. e.* supported merely at the two extremities, viz. at v, by the thrust of the equal and opposite portion; and at the lower surface ABC, by the reaction of the work from which it springs.

Let *abcfed* be an element, whose upper and lower surfaces are perpendicular to the interior surface of the dome.

The weight of this element is proportional to its solid content, *i. e.* to the continued product of its three linear dimensions. Wherefore, putting t, for bc; ds, for af; and x, the distance from the axis, for ab, to which it is proportional; we shall have

$$t.x.ds = \frac{\text{c}}{r}.\ \sec.^2\theta.ds. \qquad \text{or, } t = \frac{\text{c}}{r.x}.\ \sec.^2\theta.$$

At the crown, x is cypher; and at the base, sec. θ is infinite: wherefore, at the crown and at the base the thickness is infinite, unless r is also infinite at those points.

If the dome is a segment of a sphere, r is constant; and sec.θ is $\frac{r}{y}$; y being the ordinate on the horizontal diameter of the sphere. Wherefore,

$$t = \frac{\text{c}.r}{x.y^2}.$$

The thickness t is a minimum, when $x.y^2$ is a maximum, *i. e.* when $x = \frac{r}{\sqrt{3}}$. which corresponds to a distance from the summit of 36 degrees, nearly.

11. From the thickness required at the top, it appears that an equilibrated arch consisting of two sectors of a dome, meeting at their highest points, cannot be constructed, unless by making the curvature at the highest point infinitely small: and it might be supposed, that the same limitations belong to a dome consisting of apposited sectors. But in a dome there are other means of security, arising from the thrusts in each horizontal course. The nature of these horizontal thrusts will be understood, by conceiving the dome to be open at the top. The opening being a horizontal circle, the highest course is an arch, and the tendency of any part to fall inwards, being equable throughout, is effectually counteracted by the mutual pressures of the parts. Moreover, whilst the parts of each course are prevented

from falling inwards on the principle of the arch, the inward pressures, whilst permitted to subsist, are fitted to resist any accidental violence tending to push them outwards; and being of the nature of compressing forces, to bind all together into one compact mass.

Such is the advantage to be derived from the inward pressures in the horizontal courses. But in a dome consisting of equilibrated sectors, these sectors being merely apposited, do not press against each other. The pressures inward, in each horizontal course, are therefore counterbalanced by the thrusts from above and below. Accordingly, a dome consisting of such sectors, having no such security against any violence tending to push any part outwards, is the weakest of all domes, which can be made to stand independently of the adventitious aid of hoops, cramps, cement, and friction.

In a dome which swells outwardly beyond that composed of equilibrated arches, not only is the inward pressure in the horizontal courses removed by the thrusts directed from above and below, but there is moreover a tendency to fly outwards; and as there remains nothing by which this is resisted, the dome must necessarily fall, by bursting outwards.

But in a dome, which falls within that composed of equilibrated sectors, the tendency inward in the parts of the horizontal courses is never fully removed; and in a structure of this kind, outwardly concave, this tendency is even increased by the pressures from above and below: for which reason, such domes are much stronger than those consisting of equilibrated sectors.

From hence it would appear, that a convex dome may be strengthened by removing part of the crown: for the outward thrusts from above and below are thereby diminished; whilst the upper course, by the tendency of its parts inward, constitutes an arch which serves as an abutment for the

portions of the sectors beneath. A dome open at top may, therefore, carry a cupola or lantern; provided that the weight of the superstructure does not exceed that by which the opening should be filled up, to render it a dome of equilibration.

It is true, that the tendency to burst outwards, may be counteracted in a dome of any figure, by hooping, &c. and the horizontal thrusts, moreover, taken from the walls of the building over which it rises, by the same means. When this method is not taken, the massiveness of the walls must be suited to the thrust they have to sustain: and it appears from Art. 5. that, in an equilibrated dome, this is equal to the weight of a portion of the dome, measured from the summit to the course, whose upper surface is inclined to the vertical in an angle of 45°.

SECTION X.

OF STRENGTH AND STRAINS, DIRECT AND TRANSVERSE; AND OF THE PRINCIPLES OF FRAME WORK.

1. When a soft body, of a uniform substance, is compressed by a force, equally applied to every part of its surface, it is not broken. Hence it follows, that fracture is produced by a compressing force, in consequence of the inequality of its actions, or of the forces of cohesion, by which it is resisted.

Bodies are crushed by the lateral detrusion of the parts from their places: and this is variously effected, as the bodies are malleable or rigid, of a uniform, fibrous, or granular structure.

If the body is malleable or ductile, it is easily conceived, that of the particles situated in the direction of the compressing force, some shall be detruded, laterally, from between others, without breaking the continuity of the body: and if this operation of the compressing force were equable throughout, the length of the body would be diminished, and the breadth uniformly increased. But this is not the case; for if the compressing force is evenly applied to the circular ends of a column of such a substance, the detrusion or swell shall be greatest where the resistance is least, *i. e.* at the ends, as is represented by the toruses of architectural columns; the force of resistance being there least, in relation to the compressing force. If the body is rigid, and still homogeneous, it shall begin to fail, by splintering off in the same parts.

To find the angle of the splinter, let *acef* be a pillar of a

homogeneous substance, loaded with a weight w, (Fig. 109.) and let *aob* be the angle of the fracture, to be now determined. The cohesive force in the section *ob*, or *od*, varies as the lengths of these lines, *i. e.* inversely as the cosine of the angle *aob*, or *cod*, *o* being the middle point of the line *ac*. The incumbent weight, w, is a force applied in the vertical; and, therefore, its energy in forcing the portion *aob* along the inclined plane *ob*, is proportional to the sine of the same angle: wherefore, the resistance divided by that energy, varies inversely as the product of the sine and cosine of the same angle; *i. e.* as $\frac{1}{\sin. 2aob}$. and as the body will fail where the resistance divided by the energy of the weight is a minimum, it will fail where $\sin. 2aob$ is a maximum, *i. e.* at an angle $aob = 45°$. Wherefore, $bod = 90°$. and a pillar of iron is said to fail in this way, under enormous pressure; the surfaces of fracture making a right angle with each other, as represented in the figure.

A fibrous substance will not always follow this rule; for if the direction of the compressing force is parallel to that of the fibres, and these slightly agglutinated, the fibres may give way by bulging in the middle rather than at the ends, where the lateral motions are resisted by enormous friction. In this case the outer fibres, having least support against the force by which they are bent, will be most detruded, whilst those nearer the axis are crippled, as is represented in (Fig. 110). This change continues until the mutual cohesion among the fibres is broken, and then the body is crushed. A pillar of such material under a compressing force, is greatly strengthened by hooping.

In a uniform substance, the strength to resist compression is generally far greater than the strength to support an extending force. But in fibrous substances, it is otherwise: for the extending force, having the effect of uniting the fibres, cannot break the cohesion without overcoming their

united strength. However, in the relations of their strength to bear a load suspended or incumbent, there is the greatest diversity. Thus, a beam of oak will suspend twice as much as an equal beam of fir; whilst it will support but one-half. The cause of this diversity is probably to be found in the curvature of the fibres of the oak, which renders it less fit to support than to suspend.

2. But the cohesive forces, which constitute the strength of a beam or rod, are more easily surmounted by a force applied transversely; especially, if the straining force is applied at a considerable distance from the point of fracture. This is plainly to be attributed to the advantage of leverage then given to the straining force; as also, perhaps, to the circumstance that the full force of cohesion in the section of fracture is not at once encountered by it.

The theory of transverse strains is not as yet well established, owing to the difficulty of ascertaining the physical conditions of the bodies subjected to those actions. Galileo was the first who attempted to bring this matter within the province of theoretic mechanics; and little has been added to what he had delivered on the subject. Some improvements, however, have been made in this theory, by Mariotte, Varignon, and Leibnitz.

The operation of a transverse strain is to separate two surfaces, before in contact, by an angular motion round a certain line, which then becomes an axle or fulcrum; and it was supposed by Galileo, that this line was in that surface which is rendered concave by the action of the straining force. It was also supposed by him, that the force by which the motion round this line is resisted, is the same in all the fibres by which the divided surfaces had been connected. On this supposition, the sum of the forces of the fibres might be supposed to be applied at the centre of gravity of the transverse section, which is the surface of fracture; and the leverage of this force of resistance would, therefore, be the

distance of this point from the axis of rotation. According to this account, the moment of the resistances in a rectangular beam, would be the sum of the cohesive forces of the fibres in the section of fracture, multiplied by half the depth of the beam, *i. e.* f denoting the cohesive strength in a superficial unit of the transverse section; b the breadth; and h the depth of the beam; the sum of the cohesive forces would be $f.b.h.$ and its moment $\frac{f.b.h^2}{2}$.

Hence it appears, that the strength of a rectangular beam to resist a transverse strain, varies as the breadth multiplied by the square of the depth: for which reason, such a beam placed horizontally is stronger to resist a vertical strain when its narrow face is uppermost, than when it is placed with its broader surface in the same position, in the inverse ratio of these dimensions: so that if the breadth is double of the thickness, the strength in the two positions will be as two to one.

The strongest rectangular beam that can be cut out of a given cylinder is that wherein $b.h^2$ is a maximum. This gives

$$2b.h.dh + h^2.db = 0.$$

or,

$$2b.dh + h.d.b = 0.$$

Also, r being the radius of the cylinder, we have

$$b^2 + h^2 = 4r^2.$$

whence

$$b.db + h.dh = 0.$$

and db being exterminated by these two equations, we have

$$h^2 = 2b^2. \quad i.\,e.\ 3b^2 = 4r^2.$$

whence we derive the following simple construction.

Let AzBo be the circular section of the cylinder; and AB its diameter, (Fig. 111.) Let this diameter be divided

into three equal parts by the points m and n: then erecting the perpendiculars mz, no, and drawing the lines zA, zB, oA, oB, we have the rectangular section required.

3. The supposition relative to the equality of the forces exerted by the fibres in the section of fracture, was first contested by Leibnitz. It was properly observed by him, that as all beams were more or less deflected before the instant of fracture, the fibres could not be equally strained; and that as the forces exerted are known to be proportional to the extensions, those forces of resistance in the several fibres must be proportional to their distances from the fulcrum, or centre of angular motion. Thus, let ab, ac, be two lines drawn in the two faces separated by fracture, (Fig. 112.) those lines being perpendicular to the axis of rotation passing through the point a; we may then confine our attention to the strains in the triangular surface abc. The strength of a fibre to resist a longitudinal pull being denoted, as before, by f, this will be the force exerted by the fibre bc, when about to break: and that of any other fibre will be less in the ratio of its distance from a to the depth of the beam. Accordingly, the sum of the forces of resistance, exerted at the moment of fracture, would be only $\frac{f.b.h}{2}$, instead of $f.b.h.$ which would have been the measure of the strength if all the fibres acted alike, according to the Galilean hypothesis. Moreover, the sum of these strains would be applied at the centre of gravity of the triangle abc. whose distance from a, the centre of motion, is $\frac{2}{3}h$. and accordingly, the moment of the forces of resistance would be measured by $\frac{f.b.h^2}{3}$. instead of $\frac{f.b.h^2}{2}$.

It is be observed, that the conclusions obtained from the Galilean hypothesis as to the comparative strength of a rectangular beam, in different positions, would have equally

followed from this of Leibnitz; but this agreement does not extend to all figures.

Thus, if a triangular beam, with its end fixed in a wall, were to support a weight suspended from the other end; according to the Galilean hypothesis, the weight which it would support, with its base uppermost, would be twice as great as that which it would support with its edge uppermost; the distances of the centre of gravity of the triangular section from the fulcrum being, in the one case $\frac{2}{3}h$, and in the other $\frac{h}{3}$. According to the hypothesis of Leibnitz, the beam, with the base of the triangle uppermost, would have a force of resistance whose moment is $\frac{f.b.h^2}{4}$. *i. e.* the area of the triangular section being $\frac{b.h}{2}$, the moment of the resistance would be that of the full cohesive force of all the fibres in this section, acting at the distance $\frac{h}{2}$ from the fulcrum. Whilst the moment of resistance in the same beam having the edge uppermost, would be $\frac{f.b.h^2}{12}$. or that of the direct cohesive strength of all the fibres in the section, acting at the distance $\frac{h}{6}$, from the fulcrum. So that according to this hypothesis, the beam in the former position would be thrice as strong as in the latter.

This diversity, in the conclusions as to the relative strength of a triangular beam, in these two positions, might seem to offer a test of the truth of the hypotheses from whence they follow. But there are other effects of inflection, which are not to be overlooked.

4. James Bernouilli was the first who proposed to consider the compression produced in the concave surface. For,

as he remarked, the whole of the fibres in the section of fracture had not been in a state of tension during the operation; but whilst on one side they were in this state, those on the other side were compressed. These portions of the sections of fracture are separated by a line transverse to the length of the beam; in which line, the fibres are neither compressed nor extended; and this line is, therefore, to be regarded as the real axis of rotation. The moment of the forces which resist extension on one side of this line, must be equal to the moment of those which resist compression on the other; for it is this equality of the moments which determines the position of the neutral axis. Were the moment of the resistance to compression the lesser of the two, the body would yield by compression, whereby the neutral axis would be further removed from the concave side, and this motion would continue until the equality is restored: and we are to reason in the same way, if the moment of the resistances to extension were the lesser. Hence it follows, that the entire of the moment of resistance is twice that of the forces which resist extension; and therefore, that the more compressible the body, the less the force of resistance which it is capable of opposing to a transverse strain. For the moment of this force is twice the moment of the forces which resist extension; and this latter moment varies as the square of the distance of the convex surface from the axis of rotation, which is the neutral axis. Hence it is that a beam of soft timber is much strengthened by a cut on the side, which, by yielding, would become concave, the cut being filled with a wedge of hard wood. For the fulcrum, or axis of rotation, is thereby removed to the surface, and what is lost by the absence of the forces which resist compression, is more than supplied by the increased leverage given to the forces which resist extension. Thus, if the neutral axis is supposed to be in the middle of the section of fracture, and the saw-cut to be half the depth of the beam,

the distance of the centre of gravity of the uncut half, from the axis of rotation, becomes thrice as great, and therefore, the moment of the forces, excited by extension, is increased in the same ratio: wherefore, according to the Galilean hypothesis of the equality of the forces excited in the fibres, the moment of resistance would be increased in the ratio of 3 : 2.

To see what would be the increase of strength according to the hypothesis of Leibnitz, we have only to integrate the expression $f\frac{b}{h}x^2.dx$ from $x=\frac{h}{2}$ to $x=h$, and this would give $\frac{7}{24}f.b.h^2$ for the moment of the resisting force; whereas the sum of the moments of the forces resisting extension and compression before the introduction of the wedge is $\frac{4}{24}f.b.h^2$. so that according to this hypothesis, the strength would be increased in the ratio of $3\frac{1}{2}$: 2. This is on the supposition, that the compression can be altogether prevented in this way. But, as this is only in part prevented, we are not to expect an increase of strength to the amount of what is here inferred.

5. Two important changes in the theory, as delivered by Galileo, have been already noticed; and it does not appear that, with these emendations, the theory comprises all the physical conditions which influence the result. It has been judiciously remarked by Mr. Barlow, that the fibres do not break immediately after being strained to the utmost of their strength; but that after this, the force of resistance still continues to act, though with decreasing energy; whilst other fibres, nearer to the axis of rotation, exert their greatest force: and, accordingly, that fracture does not occur until the aggregate of the moments of the forces of resistance shall have passed its maximum. But when this occurs, *i. e.* how far the exterior fibre is stretched, before that the whole begins to give way, cannot possibly be known, unless the law of the decreasing energy of a fibre, after exerting its greatest

force, were ascertained. To this may be added another observation, which is a consequence of the preceding; that after the exterior fibres on either side have exerted their maximum of force, whether in resisting extension or compression, the neutral axis must recede from that towards the other side, where the fibres have not as yet been strained or compressed to the limit of their strength. This motion of the neutral axis will be towards the side, to which it had been originally nearest, the weaker forces being those first overcome. But it does not follow, that the beam shall be broken by the weight which weakens the strength on one side only, for this may occur before that the fibres at the other side have arrived at the limit of the forces which they are capable of exerting. Thus a beam is often crippled at one side, by a force which will not break it. These observations have been made for the purpose of showing that it is in vain to expect a complete theory, or one that shall perfectly accord with experiment, until the mechanical process towards fracture is better understood. In the mean time, the theory, as far as it has been carried, suggests many useful hints to the practical mechanic. Thus it was shown, why a beam of a compressible substance is strengthened by the insertion of a hard wedge into a cut made in the concave side; and the same theory suggests the best methods of fishing and splicing. It appears also, that the strength of a joist may be increased, whilst the material is economized, by increasing the depth in relation to the breadth: and that a similar advantage may be attained in the construction of metal pillars, by casting them hollow. For a hollow cylinder is stronger than a solid one of the same substance; the centre of forces opposed to fracture, being further removed from the fulcrum in the hollow than in the solid cylinder, whilst the surface of fracture is the same in both. There is however a limit to the extent to which this principle should be carried: for as all the parts should act together, they should

be left sufficiently strong to resist such impressions as would otherwise cause a partial violation of figure.

All this is seen exemplified in the works of nature, where lightness is not less requisite than strength; as in the bones of most land animals, the quills of birds, the stems of reeds and grasses: and suitably to the purposes of the Allwise Contriver, we find that the substances of such bodies are of a more than ordinary degree of hardness and tenacity.

On the whole, it is to be observed, that the equilibrium between the straining and resisting forces continues until the moment of fracture. For as the former is increased, the resisting forces are also more intensely excited until their moment becomes a maximum; and it is only when the straining force is increased beyond this limit, that the equilibrium is violated.

6. Hitherto the strength of materials to resist fracture has been considered, chiefly according to the different modes in which this strength may be supposed to be exerted: but it is no less necessary to form an estimate of the efficacy of the straining forces, according to the different modes in which they may be applied. In treating of this part of the subject, the energy of the resisting forces may be regarded as unvaried.

If a beam, having one end fixed in a wall, carries a weight on the other end, the whole may be regarded as a bent lever of the first kind: the fulcrum being at the axis of rotation, in the plane of the surface in which the beam would break when overloaded. The arms are the distances from this fulcrum to the point at which the weight is applied on the one hand, and to the centre of the resisting forces on the other. And in the case of equilibrium, these contending forces are reciprocally as the lengths of the arms by which they act; or, more properly, they are reciprocally as the perpendiculars let fall from the axis of rotation on the directions of the forces. In what follows, the flexure of the beam

being supposed to be inconsiderable, the length of the arm shall be taken as the measure of leverage.

Whilst the beam is secured at one end, the energy of a weight, w, to produce a strain in a transverse section at the distance x, from the weight, is $w.x$. For the equilibrium subsisting between the straining force, and that by which it is resisted in any section, will not be disturbed if the beam is fixed at that section: the weight will then act, by the arm of a rigid lever, against the resistances in the same section. The energy of the weight, being expressed as above, is greatest in the section which coincides with the plane of the wall: and putting l for the length of the beam, or the distance of the weight from the surface of the wall; its value at that distance is $w.l.$

7. If a beam, supported loosely at its two ends by props, carries a weight at any intermediate point, the strain on the section made at the part where the weight is applied, is evidently the same as if it were there placed on a wall, and loaded at each end with a weight equal to the reaction of the prop which supports that end. Therefore, to find the energy of the strain, we have only to find the pressure on each of the props, and to multiply it into the distance of that prop from the weight. Accordingly, p, p' denoting the pressures on the props; A and B (Fig. 113.) d, d' their distances from the weight at C, we have

$$p = \frac{w.d}{l}.$$

and for the moment of the straining force,

$$\frac{w.d'.d}{l}.$$

For any other point between A and C, whose distance from A is x, the moment of the straining force is $\frac{w.d'.x}{l}$. This moment is, therefore, greatest in the section where the

weight is applied. And if the place of the weight is varied, it is greatest when the weight is placed at the middle point of the beam, its value being then $\frac{w.l}{4}$. The reaction of the prop B has the same effect; wherefore, $\frac{wl}{2}$, the energy of the weight, applied at the middle of a beam loosely supported, is equal to that of the same weight applied at the end of a beam of one-half of the length, the other end being secured. Or, it is the same as that of one-half of the weight applied at the end of the entire beam, the other end being fixed. Whence it follows, that a beam supported loosely on props, would carry on its middle point, twice the weight which it could support at one end, the other end being fixed: and it may, therefore, be said to be twice as strong when treated in the one way, as it would be in the other.

8. If a beam were secured at both ends, it becomes still stronger to support a weight in the middle, in the ratio of $2:1$. For the weight required to produce the strain at the middle, and those at the two ends, is equal to the sum of those which would be requisite to produce them separately, *i. e.* it is the sum of the weight, which would produce the strain in the middle, the ends being loosely supported, and of the two weights which would produce the strains at the wall, the beam being sawn through the middle. Now if the three strains were equal, the moment of the weights which by acting on the half lengths produce those at the wall would be, each of them, equal to $\frac{wl}{2}$; and, therefore, to produce the triple fracture, the moment of the weight in the middle should be $\frac{3}{2}wl$. But the deflections at the ends near the wall are, each of them, only half of that at the middle. Wherefore, the weight supported by the sum of the resistances at the ends, is only equal to the weight supported by the resistance of the middle section. Consequently, by

fixing the ends of the beam in the wall, the weight which it will support at the middle is doubled.

If equal weights are uniformly distributed over a beam fixed at one end, the strain on that end shall be half of what it would have been, had the sum of the weights been applied at the remote end.

For, let dw be the weight resting on dx, a portion of the length, we shall have

$$d.w = \frac{w.dx}{l}.$$

and putting x for its distance from the fixed end, the energy with which it acts on that end is

$$\frac{w.x.dx}{l}.$$

and the sum of these moments, or

$$\frac{w}{l}.\int xdx = \frac{w.x^2.}{2.l}$$

which needs no correction. This, for $x = l$, is $\frac{wl}{2}$. which is half the moment of the weight w, when applied at the outer end.

If the beam is supported loosely on its ends, the strain at the middle point, on the same supposition of the equal distribution of the load, is the half of what it would have been, had the sum of the weights been applied at that point.

For the force on dx is $\frac{wdx}{l}$; and its moment to produce fracture at o, the middle point, is had by multiplying this into x, its distance from o. Wherefore, it is $\frac{w.x.dx}{l}$: and the sum of these from o to A, is $\frac{wl}{8}$: and the same for the weights between o and B. Accordingly, the whole mo-

ment at o, of the weight thus uniformly distributed is $\frac{wl}{4}$: which is half the strain produced by w placed at that point.

Since the energy of the uniformly distributed weights, to produce fracture, at the middle point of the beam when loosely supported, is half of that, which would belong to those weights if concentred at that point; and since, when the beam is fixed at the ends and sawn in the middle, the energy of the same weights to produce fracture at the fixed ends, is one-half of that which would have been exerted, had they been concentred at the same point; it follows, that the energy to produce both of these effects conjointly, must be one-half of that, with which the weights would have acted, if concentred at the middle point, *i. e.* if a beam is secured at both ends, and uniformly loaded, it will bear double the weight which it could have borne at its middle point.

Therefore a beam, whether fixed at one end or loosely supported at both ends, or fixed at those points, is capable of bearing twice the load, uniformly distributed, that it could have borne if placed altogether at the unsupported end of the beam in the first case, or at the middle in each of the two other cases; *i. e.* in all cases, the strength of the beam is doubled by the uniform distribution of the load.

9. If the force by which a beam is in danger of breaking is its own weight, the same theorems will apply. And it is worthy of note, that as the energy of the uniformly distributed weights varies as $w.l$, and that of the force of resistance as the breadth of the beam, multiplied by the square of its depth, *i. e.* as $b.d^2$. so if w is the limit of the weight which it will carry in this manner, it shall vary as $\frac{b.d^2}{l}$. which, for similar beams, is as l^2. *i. e.* the strength of

similar beams, estimated by the load which they will carry in this manner, varies as the squares of their lengths.

But though the strength of a beam to support an extraneous load, increases as the square of its linear dimensions, those dimensions being increased in the same proportion, yet its strength to support its own weight, so far from being increased, is diminished. The reason is, obviously, that its own weight increases as the cube; and the moment of that weight, as the fourth power of its linear dimensions, and therefore, faster than its strength to support it. Thus if there are two similar beams, whose lengths are l and l', secured each at one end in a wall: their weights are as l^3 and l'^3, and the energies of those weights to produce fracture, as l^4, l'^4, whereas, the energies of the forces of resistance are as $b.d^2 : b'.d'^2$, *i. e.* as $l^3 : l'^3$.

If the weight is known which a beam, fixed at one end, will just carry at the other end without breaking, we may find the length of a similar beam which shall just support its own weight.

For, let l be the length of the smaller beam, m its mass, and w the weight which it will just carry at the end. And designating the length and mass of the larger beam by the capitals, the energies of their loads are

$$\left(\frac{m}{2} + w\right) l. \qquad \text{and } \frac{\text{M}}{2}.\text{L}.$$

And the energies of the forces of resistance are as l^3, and L^3. And the former must be proportional to the latter, or

$$\frac{m + 2w}{l^2} = \frac{\text{M}}{\text{L}^2}.$$

Moreover,

$$\text{M} : m : : \text{L}^3 : l^3.$$

and therefore,

$$\text{M} = \frac{m.\text{L}^3}{l^3}.$$

which value of M, being substituted in the preceding equation, it becomes

$$(m + 2w)\, l = m.\,\text{L}.$$

giving

$$\text{L} = l + \frac{2\,w.l}{m}.$$

Hence it appears, that of all similar beams of the same material, and fixed in the same manner, there is but one which can barely support itself without breaking. A beam larger than that here estimated, will be broken by its own weight. And hence also it appears, that small animals are much stronger than large ones to carry their own weights; for which reason, they are far more active. Accordingly, a fly will carry ten times its own weight, perhaps with more facility than a horse can carry a weight merely equal to his own. There is, therefore, a limit to the size of animals and plants, at which they would be oppressed by their own weight, and fall to pieces. This is a natural limit to the magnitude of trees and land animals, which is further reduced, by the necessity of leaving them strength for other purposes, beyond what would be required for the support of their own weights. This limit is not so much straitened for sea animals, the weight of whose bodies is in a great measure supported, and whose strength to bear it much assisted by the weight and compression of the medium in which they live. Accordingly, we find that some of these are much larger than any of the inhabitants of the dry land.

The same principle which limits the size of the productions of nature, is equally applicable to the structures of art. Thus, what appears strong in the model, may be very deficient in strength when executed: and this must be allowed for by engineers and architects, who would, otherwise, make the most ruinous mistakes.

10. Such are the principles on which the strength of beams is to be computed, with regard to the loads they are intended to bear. In treating of the equilibrium of structures consisting of such materials, these strains were not considered. The component parts were supposed to be rigid and unbending: in which case, the loads supported by them might be supposed to be transferred to the angles: and the subject of investigation was the equilibrium of the longitudinal thrusts. When this equilibrium is secured, a great object is attained: the stability of the structure being then easily provided for. But this is not sufficient; for the parts of the structure are strained transversely by their own weights, and by the loads they carry: and it has been shewn, that the strength of beams to support such strains is far less than to support a longitudinal pull or thrust; the absolute force of resistance being much less in the former case, and acting ordinarily at a great mechanical disadvantage in relation to the straining force. Accordingly, after the equilibration of the component parts of a structure, the next object, especially in carpentry, is to convert the transverse strains into longitudinal pulls or thrusts. How this is to be accomplished may appear by one or two familiar examples.

Let AB be a horizontal beam attached to an upright at B, destined to support a weight w, appended at A, (Fig. 114). Putting l for AB, the measure of the energy of w, to strain the beam at B, is $w.l$: and unless the beam is very strong, and well secured at B, it will be broken by an inconsiderable weight, or forced out of its birth at B. This is prevented by changing the straining force to a thrust in the direction of AB: which is accomplished by combining with the force w, another force, such that the resultant of the two shall be directed according to AB: and this being effected, the position of the beam is secured, though it should be connected with the upright only by a hinge at B. Let AM be a cord

attached to the beam at A, and to the upright at M; if this is inextensible, and sufficiently strong, it must necessarily supply the force requisite to change the vertical strain into a horizontal thrust: for the point A cannot descend without stretching or breaking the cord, and if this does not occur, the three forces acting at A must be in equilibrio. These forces are, 1st, the weight w, acting in the vertical; 2d, the reaction of the cord in the direction AM; and 3d, the reaction of the upright in the direction BA. Accordingly, the two first must compound a force equal and opposite to the last, *i. e.* a force acting in the direction AB. In this case, the resultant in the direction of AB is a compressing force; and the piece AB is called a strut. We have, therefore, only to connect the point A, by a tie, to a point in the upright, above B, and the lateral strain on AB shall be converted into a longitudinal compression.

The quantity of the strain endured by the cord, and of the thrust against the upright at B, are compared with the weight w, by the triangle AMB, whose sides are in the directions of these forces. Thus, putting S for the strain on the cord, and T for the thrust against the upright, we have

$$\mathrm{S} = \frac{w}{\text{sin. ang. A}}. \qquad \mathrm{T} = \frac{w}{\text{tan. ang. A}}.$$

Similarly, by a force acting in the direction AM′, the transverse strain may be converted into a longitudinal pull. This force may be supplied by a brace AN, which, if it is incompressible, will prevent the point A from descending. The forces acting at A are then in equilibrio, *i. e.* the weight w, and the reaction of the brace in the direction of NAM′, will be a compound force in the direction of BA, which, if the hinge or fastening at B is sufficiently strong, will be counteracted by the reaction of that point. The relative magnitudes of the three forces are determined as in the former case.

In the same way a beam may be strengthened, by what, at first view, might appear to be an addition to its load. Thus, let the beam be AB, resting on supports at A and B, (Fig. 115). This shall be greatly strengthened by the piece DC, descending from its middle point, and connected by its remote end C, with the extremities of the beam AB. For the point D cannot descend without crushing the piece DC, or breaking the connexions AC, BC. Therefore, if these parts are sufficiently strong, the strain on the beam at D, where it is most dangerous, is transferred to the points A and B where it is firmly supported. In this case, the connecting pieces AC, BC, serve to thrust the piece CD against the middle of the beam: wherefore, this last is a strut, and the two former ties; for which reason, these may be inextensible cords. In this manner, the transverse strains are removed by ties and struts.

11. To take away the lateral thrusts from the walls of a house, occasioned by the weights of the rafters and their incumbent load, the heads of the rafters are connected by a tie beam; which, if it had no other office, needs not be gross. But, commonly, it has a ceiling to support, and perhaps a floor with all the furniture of an apartment. In this case, the tie beam, unless supported, will be apt to yield to the transverse strain, occasioned by its own weight and that of the load it carries. This support is given to it by a tie, or king post, connecting it in the middle with the summit of the rafters. But the rafters also, if long, will require support against the transverse strains, to which they are subjected. This is supplied by braces, descending from the middle of the rafters, and abutting against the lower end of the king post, on which they are joggled.

In this way, the middle of each rafter is supported; and the load and its bearing being reduced each to one-half, the strain upon the half rafter is reduced to one-fourth of that on the entire rafter supported only at the ends. All this is

exhibited in (Fig. 116.) where AB, AC are the rafters; BC the tie beam; AD the king post; and ED, FD the braces. This is the most common kind of roof, and it is sufficient for all ordinary purposes.

12. But in trusses and frame work of all kinds, besides the equilibration of the longitudinal thrusts, and the removal of the transverse strains, it is also requisite to provide for the inviolability of the figure, against the various influences to which the work may be exposed. When this is secured, all the parts shall act together as one solid mass; each contributing to support the strains on the rest. This end is attained chiefly by resolving the whole into triangles. For in such figures, whilst the lengths of the sides continue unchanged, their relative positions must also be unchanged. This is not the case in figures consisting of any greater number of sides. For example, nothing could be worse than a trapezium roof consisting only of three rafters and a tie beam, such as that represented in (Fig. 117). For though the thrusts are taken from the walls, by the tie beam, and the points A and B thereby fixed, yet the angles being changeable, the strain on each piece from its own load, and also, the thrusts against it from the contiguous members, are, consequently, changeable.

Were the middle points of the horizontal beams connected together by the piece MN, (Fig. 118.) those beams would be constrained to act together; but the deflections to which they are liable being in the same direction, little is thereby gained, in proportion to the quantity of material added. Were the middle points of the rafters connected together by the piece OP, the deflection in those parts would be resisted somewhat more effectually, for should the rafters bend, it must be in the same direction; but the strains from their loads being both inwards, *i. e.* in opposite directions, they would yield only by the difference of those strains.

By means of the two uprights, CE, DF, the roof becomes considerably stronger: (Fig. 119.) for the angles of the triangles, ACE, BDF, are unchangeable as long as the distances of their angular points remain unchanged. And so are also the angles of the four-sided figure CEFD, as long as the lengths of its sides, and those of the lines AC, BD, are unchanged. But the chief recommendation of this construction is the space CEFD, which it allows for an apartment. For the rafters AC, BD, the truss beam CD, and the tie beam AB, are all liable to deflection from the transverse strains.

These deflections are effectually prevented by continuing the rafters above the truss beam, so as to meet at a point; or should this roof be deemed too high, the pitch of the rafters above the truss beam may be diminished, as represented in (Fig. 120). For thereby a new fixed point is gained, by means of which, the truss beam may be prevented from bending by the king post HM. This gives another fixed point, M, which may serve for abutments to the braces MN, MP, by which the stiffness of the upper rafters is secured. The triangle CHD, being then a firm piece of frame work, C and D are fixed points, from which the queen posts, CE, DF, may descend, to carry the tie beam AB. Two more fixed points, E and F, are thus obtained, which serve as abutments for the braces EI, FK, by which the rafters AC, BD are supported. The braces EG, FL may be added to counteract the push of the queen posts, which might be occasioned by the thrusts of the braces IE, KF. They also afford additional security against any change in the angles C and D; and also, further support to the truss beam CD.

A roof consisting of four rafters of this external form is called a Mansard or kirb-roof.

13. Where firmness is the main object, the frame work should consist of triangles. For the angles being unchangeable whilst the lengths of the sides are unchanged, the figure cannot be violated except by the deflections of the

beams: and the same angles afford the means of security against these deflections, being so many fixed points, which may serve for abutments to the braces, by which the transverse strains are supported. But where it is particularly desirable to gain internal space, the angles of a kirb-roof may be secured by braces dividing each of the rafters into three parts. In this way the sides of the polygon are doubled, as represented in (Fig. 121). And by a continuance of the same method, they may be further multiplied, until the interior figure approaches to the form of a vault; it may then be lined so as to perfect that figure.

But a most ingenious method of framing a roof, without encroaching considerably on the space within, is that practised by the Normans, in what are called Gothic Buildings. The construction is that represented in (Fig. 122.) where BE, CF, DG, are pieces descending from the angles made by the rafters, the ends of those pieces being connected with the angles at each side. Thus each four-sided figure, as ABCE, is fixed, being resolved into two triangles by its diagonal BE. The whole being treated in this manner, the rafter AB cannot move independently of BC; nor BC without CD; nor this last without DE. All must, therefore, act together in resisting a force applied to any one part. Each of the braces, EA, EC, &c. is alternately a strut and a tie; a strut in upholding the angle from whence it proceeds; and a tie in holding up the piece descending from the neighbouring angle, by which means that angle is supported. Thus, CE, CG are struts, rising from the abutments E and G, to support the angle C; and CG, GE are ties upholding the piece GD against the angle D.

The same method is extended to support the rafters also against the strains occasioned by their respective loads, and this without encroaching so much on the space within; simply by connecting the pieces BE, CF, DG with the middle points of the rafters, as represented in (Fig. 123). In this

way the braces IF, KF are struts abutting on F, and supporting the rafters CB, CD at their middle points I and K. And the same pieces are ties pressing up the piece FC against the angle C.

The architects who used these methods, instead of concealing the roof by a ceiling, frequently exposed it to view, ornamented with carved work: vain, perhaps of their skill in supporting an enormous roof by that which, to a common beholder, would seem an additional load, fitted only to hasten its ruin.

SECTION XI.

OF THE ELASTIC CURVE.

1. In what has been hitherto delivered relative to transverse strains, the forces considered were those requisite to produce or resist fracture; but by a force less than this, a beam or rod is sensibly inflected. Sometimes this change of figure is permanent; in which cases, the rod is said to have taken a set: but often, also, it regains its primitive figure, when released from the straining force, in which case, it is said to be elastic; and the force, by which it is brought back to its primitive position, is called its elasticity.

When an elastic plate, fixed at one end, is bent by a force applied at the other, the figure which it assumes is called the elastic curve; and the purpose in the present section is, to point out the mode of investigating that figure.

Let OB be the plate fixed at O, (Fig. 124.) and let this be bent into the curve OB′, by a force P, acting at B′, in the direction B′G. The equilibrium at any point, M, shall not be disturbed, by rendering the plate inflexible at every other point. To conceive the nature of the action of the elastic force, the curve formed by the convex surface of the plate may be regarded as a polygon with an infinite number of equal sides. Mm, Mm' being two contiguous sides of this polygon, the tendency of the elastic force at M, is to bring the line Mm to the same direction with Mm' by a circular movement round a certain line in the section made at M, perpendicular to the plane of the curve; which line is, therefore, to be regarded as the axis of circular motion. This tendency being counteracted by P, the straining force acting

at B′, it follows that these two forces are in equilibrio; or that their moments are equal. Wherefore, putting E for the moment of the elastic force acting at M, and drawing the line Mn perpendicular to B′G the direction of the inflecting force, we have

$$\text{E} = \text{P} \times \text{M}n. \tag{1}$$

In deriving the equation of the curve from this fundamental proposition, we shall confine ourselves to the case of a plate of uniform breadth and depth, in which the flexure is inconsiderable. Moreover, as the inflecting force when oblique, may always be conceived to be compounded of two forces, of which one is perpendicular to the face of the plate, and the other in the direction of its length, it will be sufficient to consider these two cases apart, which is what is proposed in the two following articles.

2. Let ds be any element of the interior curve; *i. e.* of the curve formed by the concave surface of the plate, and let ds' be the corresponding element of the exterior curve. In the plate before inflection these were equal; but after inflection ds' is greater than ds. Now ds', ds being regarded as similar arcs of two concentric circles, whose radii are r', r, we have the following proportion:

$$ds' : ds :: r' : r.$$

and

$$ds' - ds : ds :: r' - r : r.$$

giving

$$ds' - ds = \frac{r' - r}{r}.ds.$$

But $r' - r$ is the thickness of the plate which is supposed to be unvaried: wherefore $ds' - ds$ which is the distention of the elementary portion ds varies as $\frac{1}{r}$. *i. e.* inversely as the radius of curvature. Now E varies as $ds' - ds$ multiplied by

*

the breadth into the square of the depth, which in the case now considered are constant: it therefore varies inversely as the radius of curvature; a fact which had been previously established by experiment. Wherefore $\text{E}r=\text{A}$, a constant quantity. Accordingly, taking the point O for the origin, the right line OB, or the length of the plate before inflection, for the axis of x, and the perpendicular OC for that of y, and putting x' for OD, equation (1) becomes

$$\text{P}(x'-x)=\frac{\text{A}}{r}.$$

If the flexure is small, x' may be deemed equal to OB; which being denoted by l the equation becomes

$$\text{P}(l-x)r=\text{A}.$$

This is brought to an equation between x and y, simply by substituting for r its value, which is $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx.d^2y}$. Wherefore the differential equation of the curve will be

$$\text{P}(l-x)\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx.d^2y}=\text{A}.$$

But

$$\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx.d^2y}=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

whose approximate value is $\frac{dx^2}{d^2y}$ the flexure being inconsiderable: in which case the equation becomes

$$\text{P}(l-x)\frac{dx^2}{d^2y}=\text{A}.$$

or

$$\text{P}(l-x)dx=\text{A}.\frac{d^2y}{dx}.$$

And by integration,

$$\frac{\text{P}x}{2}(2l-x)=\text{A}.\frac{dy}{dx}.$$

which requires no correction, inasmuch as for $x=0$, there is $\frac{dy}{dx}=0$.

Multiplying the last equation by dx, and again integrating, there is

$$\text{A}y=\frac{\text{P}x^2}{6}(3l-x).$$

which as x and y simultaneously vanish, requires no correction.

For $x=l$, this becomes

$$\text{A}y=\frac{\text{P}l^3}{3}.$$

And A being constant, this equation shows that the deflection at the point to which the inflecting force is applied, varies as that force multiplied by the cube of the distance of the point of application from the fixed extremity of the plate.

3. If the direction of the force P coincides with the line connecting the two extreme points O and B, the point O needs not to be fixed; it will suffice if the rod rests against an immoveable obstacle at that point: and the case will be the same with that of a vertical rod resting on a horizontal plane, and carrying a weight on its summit.

Let BO be the elastic plate resting on the horizontal plane at O, (Fig. 125.) and bent into the figure BMO, by a weight P, applied at B. Also, MN being a perpendicular let fall from any point M, on the right line BO, the direction of the compressing force, let $\text{MN}=y$, $\text{ON}=x$. then we shall have

$$\text{P}y=\frac{\text{A}}{r}.$$

in which A is constant, being the moment of the elastic force at any point multiplied into the radius of curvature at the same point. The approximate value of r, for small inflections, being $\frac{dx^2}{d^2y}$, and d^2y being negative, the equation becomes,

$$\text{P}y = -\text{A}.\frac{d^2y}{dx^2}.$$

Multiplying both members of this equation by $2dy$, and integrating, we have

$$\text{P}y^2 = -\text{A}.\frac{dy^2}{dx^2} + \text{C}.$$

To find the value of C, let the greatest ordinate of the curve be b, and for this ordinate we shall have $\frac{dy}{dx} = 0$. and therefore, $\text{C} = \text{P}.b^2$. which gives

$$\text{P}.(b^2 - y^2) = \text{A}.\frac{dy^2}{dx^2}.$$

or,

$$dx = \frac{\sqrt{\text{A}}}{\sqrt{\text{P}}}.\frac{dy}{(\sqrt{b^2 - y^2})}.$$

Again integrating,

$$x = \frac{\sqrt{\text{A}}}{\sqrt{\text{P}}}.\ \text{arc}\left(\sin. = \frac{y}{b}\right) + \text{C}'.$$

But for $x=0$, there is $y=0$, and therefore, arc $\left(\sin. = \frac{y}{b}\right) = 0$. Wherefore, $\text{C}' = 0$. and accordingly the equation of the curve is

$$x = \frac{\sqrt{\text{A}}}{\sqrt{\text{P}}}.\ \text{arc}\left(\sin. = \frac{y}{b}\right). \qquad \text{or, } y = b.\sin.\frac{\sqrt{\text{P}}}{\sqrt{\text{A}}}.x.$$

For $x = \text{OB} = l$, we have $y=0$. and therefore,

$$b.\sin.\frac{\sqrt{\text{P}}}{\sqrt{\text{A}}}.l = 0.$$

Whence, either $b=0$, or $\sin.\frac{\sqrt{P}}{\sqrt{A}}.l=0$. and to satisfy this last equation, we must have

$$\frac{\sqrt{P}}{\sqrt{A}}.\ l=m\pi.$$

in which m is some integer number. Accordingly, if the force P does not satisfy this last equation, we shall have $b=0$. *i.e.* there can be no inflection. Hence it appears, that the least value of P, which can bend the rod or plate, is that obtained by making $m=1$, in the equation $\frac{\sqrt{P}}{\sqrt{A}}.\ l=m\pi$. which then is

$$P=\frac{\pi^2.\ A}{l^2}.$$

and the same is the greatest force which the plate can sustain without bending.

SECTION XII.

OF THE FUNDAMENTAL PRINCIPLES OF STATICS.

1. The first who attempted to raise Mechanics to the rank of a demonstrative science, was Archimedes; and by him it was founded on the principle of the lever, which he established in the following manner.

Let AB be the lever, and C its fulcrum, (Fig. 126). Taking in this line produced, BD = CA, AF = CB, the whole line FD shall be bisected at C, and if FD were a cylinder, it would be supported by a prop at C. Moreover, taking AE = AF, the line ED is bisected at B. Wherefore, if the cylinder were divided into two distinct cylinders at E, the part FE would be supported by a prop at A; and the part ED by a prop at B, *i. e.* a force equal to the weight of FE, acting vertically upwards at A, and a force equal to the weight of ED, acting vertically upwards at B, would equilibrate the cylinder as effectually as a force equal to its whole weight, acting vertically upwards at its middle point C. But the force, equilibrated by this last, is the entire weight acting vertically downwards at C. Whence it appears, that two forces acting vertically at A and B towards the same side of the line AB, shall equilibrate a third force acting at C in a parallel and contrary direction, provided those forces are as FE, ED, FD; or as the halves of those lines, viz. CB, CA, AB, each of the forces being as the distance between the points of application of the other two forces.

Various improvements in this demonstration have been since proposed, by those who have regarded the lever as a suitable foundation of mechanical philosophy. That the

whole theory of statics may be derived from thence, shall be freely admitted; but it must also be admitted, by those who are most favourable to this proceeding, that the methods of applying the principle of the lever to other matters, such as the pully, the inclined plane, the composition and resolution of forces, are forced, isolated, and circuitous.

2. The principle next applied to the same purpose, was that of the composition and resolution of forces, first introduced by Newton in his 'Principia Mathematica Philosophiæ Naturalis,' which had received the imprimatur of the president of the Royal Society, early in July, 1686, though the merit of the improvement has been claimed for Varignon. This pretension, however, does not appear to have any support besides his own assertion, as his 'Nouvelle Mecanique' did not make its appearance until the year 1725, which was after his death; and no trace of the discovery can be found in any of his other productions. It plainly appears, that it had not been made by him in 1685, the year in which he published his 'Memoire sur les poulies a moufles' in the 'Histoire de la Republique des lettres; wherein, unquestionably, he would have adopted this principle, had it been then known to him, as it appears that he was afterwards fully sensible of its immense importance; and as in that perfomance, it would have afforded him peculiar facilities in calculating the efficacy of oblique strains. The principle of the composition and resolution of forces pervades the whole of the Principia Mathematica: and as it is impossible to suppose that this performance could have been the work of one or two years, it must be admitted, that it was known to Newton long before the year 1685, when unquestionably it was unknown to Varignon; and, therefore, that Newton was the first who made the discovery, as well as the first who gave it to the public.

The demonstration by which this principle was originally supported, was founded on the composition and resolution

of motion. For taking the motions produced by the single forces to represent these forces in quantity and direction, the motion resulting from the composition of those two motions was taken to represent the resulting force. This mode of proof is recommended by its extreme simplicity. But it has been objected, that it introduces the subject of motion, which is different from that under consideration; and that it assumes the proportionality of the force to the velocity produced by it. That the velocity produced in a given body, by a force acting during a given time, is the just measure of that force, is a truth established by the most extensive experience; but it is only by experience that it can be established: whereas, by an independent proof of the principle of the composition and resolution of forces, the whole theory of statics is presented as a series of necessary truths, independent of experiment or observation of any kind. These objections were first urged by Daniel Bernouilli; and he was also the first who supplied a demonstration of the kind required. Many others have been afterwards furnished, of which the most simple and elegant was that of Duchalya, the same which, with very little change, has been adopted in Sect. I. Art. 2. of this elementary treatise.

3. But another principle could not long escape the notice of writers on mechanics, being offered to their view in all instances of equilibrium; and that most obviously in the several mechanic powers: this has been denominated the principle of virtual velocities; to explain which, it is to be observed, that a force being applied to a material point, and any small motion given to that point, by which it describes the space ds in a given time, the space ds is the measure of the velocity of that point. And if this line makes with the direction of the force, the angle θ, then $ds. \cos. \theta.$ is the velocity estimated in the direction of the force itself. Now the principle of virtual velocities is thus stated: "If any number of forces are in equilibrio, whether applied to the

same point, or to different points of a system, and any small movements are given to the system of points, such as their mutual connexions will admit, the sum of the products obtained by multiplying each force into the velocity of its point of application, estimated in the direction of the force, is cipher; and conversely, when for all the small movements, which can possibly be given to the points of the system, the sum of these products is cypher, the forces shall be in equilibrio." Thus, if the points of the system are m, m', m'', m''', (Fig. 127.) connected together according to any condition; and if by any small displacement they are carried to the points n, n', n'', n''', respectively; then $mn, m'n', m''n'', m'''n'''$ are the virtual velocities; and $ns, n's', n''s'', n'''s'''$ being perpendicular to the directions of the forces, the virtual velocities, estimated in the directions of the forces, are ms, $m's', m''s'', m'''s'''$. Now, putting P, P', P'', P''' for the forces acting at those points, and p, p', p'', p''', for the lines ms, $m's, m''s'', m'''s'''$, then according to the principle of virtual velocities, the system being in equilibrio, we shall have

$$P.p + P'.p' + P''.p'' + P'''.p''' = 0.$$

And conversely, if the condition expressed by this equation is fulfilled for every small movement which may be imparted to the system, the forces are in equilibrio.

In announcing this principle, it is to be noticed, that the forces P, P', P'', &c. are always deemed positive; and that p, p', p'', &c. are positive, when their directions are the same with those of P, P', P'', &c. and negative, when their directions are opposed to those of the same forces. Thus, in the figure; $m's', m''s''$, being measured from the points m', m'', in the directions of the forces P', P'', are positive; but $ms, m'''s'''$, which are measured from the points m, m''', in directions opposite to those of the forces P, P''', are negative.

4. It has been already observed, that this principle was discovered by induction. Thus, for equilibrium in the lever,

the power and weight acting in directions perpendicular to the arms l, l', we had $\text{P}.l = \text{P}'.l'$. But a small angular motion being given to the lever, the extreme points of the arms describe the spaces ds, ds'. And these being similar circular arcs whose radii are l, l', we have

$$ds : ds' :: l : l'.$$

Wherefore,

$$\text{P}.ds = \text{P}'.ds'.$$

But the directions of the forces being regarded as positive, whilst they tend to turn the lever about the fulcrum in opposite directions, one of the virtual velocities is positive, and the other negative; wherefore, putting $ds = p$, we shall have $ds = -p'$. which values being substituted in the preceding equation, it becomes

$$\text{P}.p + \text{P}'.p' = 0.$$

More generally, the angles made by the directions of the forces with the arms to which they are applied, being ϕ, ϕ', we had for equilibrium, $\text{P}.l.\sin.\phi = \text{P}'.l'.\sin.\phi'$. But $\frac{ds}{l} = \frac{ds'}{l'}$. wherefore, multiplying by these equals, the preceding equation becomes

$$\text{P}.ds.\sin.\phi = \text{P}'.ds'.\sin.\phi'.$$

Also, θ, θ', being put for the angles made by the directions of the forces, with the lines ds, ds', we have

$$\sin.\phi = \cos.\theta. \qquad \sin.\phi' = \cos.\theta'.$$

Wherefore,

$$\text{P}.ds.\cos.\theta = \text{P}'.ds.\cos.\theta'.$$

But,

$$ds.\cos.\theta = p. \qquad ds'.\cos.\theta' = -p'.$$

and therefore,

$$\text{P}.p + \text{P}'.p' = 0.$$

In the screw, the condition of equilibrium was found to be that expressed by the equation

$$P.2\pi r = P'.I.$$

wherein $2\pi r$ is the periphery of the circle described by the power P; and I, the distance between the threads of the screw, measured in a direction parallel to its axis: and these being the measures of the velocities of the power and resistance which, as before, are affected with opposite signs, the equation is equivalent to $P.p + P'.p' = 0$.

5. The principle of virtual velocities being thus verified by induction, it was natural that those who treated of the subject should have endeavoured to establish it by a rigorous demonstration. The proofs which have been offered are, for the most part, attempts to reduce the principle in question to that of the composition of forces. But to this mode of proceeding it is objected, that instead of making out an independent principle of mechanical science, it presents a general expression of the conditions of equilibrium deduced from other principles. Lagrange has indeed attempted to furnish a demonstration not liable to this objection, by resorting to the theory of the pully; which is readily established, independently of the composition of forces, when the parts of the cord, embracing each system of moveable pullies, are parallel. Thus, if a cord passes over any number of systems of fixed, and the same number of systems of moveable pullies, the ends of the cord being attached to the blocks carrying the extreme systems, whether of fixed or moveable pullies, and all the parts of the cord being parallel; there will be equilibrium in the entire system, when the forces applied to the several parts of the cord are equal, *i.e.* when the weights appended to the several blocks containing the moveable pullies, are proportional to the number of parts of the cord reaching those blocks. Thus, in (Fig. 128,) where in the first system there are two, in the second, three, and in the third, four moveable pullies, the extreme ends of the cord being attached to the blocks which carry

the systems of fixed pullies, there will be equilibrium when the weights are in the following proportion:

$$\text{P} : \text{P}' : \text{P}'' ::$$
$$4 : 6 : 8$$

But if either end of the cord is carried over a fixed pully, and is then attached to the lower block, the number representing the weight carried by that block is to be increased by unity; the equilibrium being established in all cases, when the weights appended to the lower blocks are proportional to the number of parts of the cord reaching those blocks; so that if n, n', n'', express the numbers of the cords which carry the weights P, P$'$, P$''$, we shall have $\frac{\text{P}}{n} = \frac{\text{P}'}{n'} = \frac{\text{P}''}{n''}$. for equilibrium. But for vertical weights, we may substitute forces equal to those weights, and acting in any different directions, provided that the cords pertaining to each of the blocks which carry the moveable pullies are parallel, as in (Fig. 129). Now if the systems of moveable pullies suffer any small displacement, let ds, ds', ds'', &c. denote the spaces described by their several blocks, estimated in the directions of the cords, and the length of the entire cord being invariable, we have

$$nds + n'ds' + n''ds'' + \&c. = 0.$$

and, therefore, substituting for n, n', n'', &c. the forces to which they are proportional,

$$\text{P}.ds + \text{P}'.\,ds' + \text{P}''.\,ds'' + \&c. = 0.$$

The application of this theorem is obvious; for the blocks being regarded as the points of a system to which the forces P, P$'$, P$''$, &c. are applied, and ds, ds', ds'', &c. being the virtual velocities of those points, estimated in the directions of the forces, we have

$$\text{P}.p + \text{P}'.p' + \text{P}''.p'' + \&c. = 0.$$

Lest the demonstration, given above, should seem to be limited to tending or divellent forces, let any one of the

moveable blocks and its corresponding fixed block be supposed to exchange places, and the demonstration shall be equally valid, *i. e.* the equilibrium shall still subsist, whilst the direction of the force, and that of the reaction occasioned by it in the fixed point, are changed into the opposite. Moreover, as the equilibrium of a system is not disturbed by changing the point of application of any of the forces for any other point in its line of direction, it follows, that the original points of application may be retained, whilst the forces there applied are converted into compressing forces.

Though it has been thus proved, that in all cases of equilibrium we have $\text{P}.p + \text{P}'.p' + \text{P}''.p'' + \&\text{c}. = 0.$ it is not to be supposed that the converse of this proposition is true, in the manner stated by some writers on the subject. The foregoing equation is indeed a test of equilibrium, when it subsists for every small motion whatever that may be supposed to be imparted to the system, and not otherwise. For, any two of the forces, P, P', P'', &c. being varied, both being increased or both diminished in the same ratio, the points of application of those forces may be supposed to be moved, the others remaining undisturbed. In the case here supposed, the virtual velocities of those points would be reciprocally as the forces there applied; and being, moreover, affected with opposite signs, the equation $\text{P}.p + \text{P}'.p' + \text{P}''.p'' + \&\text{c}. = 0.$ would still subsist, though the equilibrium no longer subsists in the entire system.

But the fundamental conditions of equilibrium in the systems of pullies are those expressed by the equations $\frac{\text{P}}{n} = \frac{\text{P}'}{n'} = \frac{\text{P}''}{n''}$, &c. and if these conditions are satisfied, the equilibrium is established. It remains then to be seen how far these conditions are secured, when that expressed by the equation $\text{P}p. + \text{P}'.p' + \text{P}''.p'' + \&\text{c}. = 0.$ is satisfied. Let this equation, therefore, be supposed to subsist: from the inextensibility of the cord, we have, moreover, the equation

$$(np+n'p'+n''p''+\&c.)\,a=0.$$

and by combining this with the preceding equation,

$$(\text{P}-na)\,p+(\text{P}'-n'a)\,p'+(\text{P}''-n''a)\,p''+\&c.=0.$$

Now, in order that this may be resolved into the equations

$$\text{P}-na=0. \qquad \text{P}'-n'a=0. \qquad \text{P}''-n''a=0. \quad \&c.$$

or,

$$\frac{\text{P}}{n}=\frac{\text{P}'}{n'}=\frac{\text{P}''}{n''}, \&c.=a.$$

it is requisite that the quantities p, p', p'', &c. should be so many independent variables. From whence it follows, that to render the condition expressed by the equation $\text{P}p+\text{P}'p'+\text{P}''p''+\&c.=0.$ sufficient for equilibrium, this equation must subsist for all the small movements which can possibly be imparted to the points of the system. But it is not requisite that the movements p, p', p'', &c. should be absolutely independent, in order that the equation $\text{P}p+\text{P}'p'+\text{P}''p''+\&c.=0.$ should become a test of equilibrium. So far as the motion of any point is restrained, whether by a surface to which it is confined, by a fixed point, or axis, round which it may revolve, or by its connexions with other points of the system, so far is the equilibrium of the system necessarily secured; whence it follows, that we may omit those movements which are impossible, and pronounce that the equilibrium is provided for, when the preceding equation is satisfied in the case of all the small movements which can possibly be imparted to the points of the system, consistently with the conditions by which those movements are restricted.

6. The principle of virtual velocities being thus explained, it remains to show, by a few examples, how it is applied.

Let there be two inclined planes, whose lengths are l, l', and whose common height is h, placed back to back: it is required to find the conditions of equilibrium for two weights, P, P′, placed on these planes, and connected by an inexten-

sible cord which passes over the summit; the parts of the cord measured from the summit being respectively parallel to the planes.

From the condition of the inextensibility of the cord, it follows, that one of the weights cannot descend, unless the other ascends through an equal space; therefore, the virtual velocities are equal and contrary. To reduce these to the vertical direction, which is that of the weights, they must be multiplied by the height, and divided by the lengths of the planes. Wherefore,

$$p = ds.\frac{h}{l}. \qquad p' = -ds.\frac{h}{l'}.$$

whence the equation $\text{P}.p + \text{P}'.p' = 0.$ becomes

$$\text{P}.ds.\frac{h}{l} = \text{P}'.ds.\frac{h}{l'}.$$

or dividing by $ds.h$, it is

$$\frac{\text{P}}{l} = \frac{\text{P}'}{l'}.$$

which shows, that for equilibrium, the weights must be proportional to the lengths of the planes on which they are placed.

Let two weights, connected by an inextensible cord, be placed on the surface of a horizontal cylinder; and let it be required to find their positions for equilibrium.

From the condition of the inextensibility of the cord, it follows, that the virtual velocities are equal and contrary; and these are reduced to the vertical, when multiplied by the sines of the arcs, measured from the weights to the summit. Accordingly, puting θ, θ' for those arcs, we have

$$p = ds.\sin.\theta. \qquad p' = -ds.\sin\theta'.$$

whereby the equation $\text{P}.p + \text{P}'.p' = 0.$ becomes

$$\text{P}.ds.\sin.\theta = \text{P}'.ds.\sin.\theta'.$$

or,

$$P.\sin.\theta = P'.\sin.\theta'.$$

which shows that the whole arc commensurate to the cord, is to be divided into two parts, whose sines are as the weights; and these parts being measured from the summit, that the greater weight is to be placed at the extremity of the lesser, and the lesser weight at the extremity of the greater arc.

The same thing would appear by regarding the weights as placed on the tangent planes; for then it appears from the preceding problem, that the weights must be reciprocally as the sines of the horizontal elevations, *i. e.* reciprocally as the sines of the arcs measured from the bodies to the highest point.

If a number of connected weights are in equilibrio, the common centre of gravity shall neither rise nor fall, in consequence of any small movement which may be given to the bodies of the system. And conversely, if such is the condition of the common centre of gravity, the weights are in equilibrio.

For the distances of the several weights from a horizontal plane being denoted by z, z', z'', z''', &c., and that of the common centre of gravity from the same plane, by Z, the equation $P.p + P'.p' + P''.p'' + P'''.p''' + \&c. = 0.$ becomes

$$P.d.z + P'.d.z' + P''.d.z'' + P'''.d.z''' + \&c. = 0.$$

But by Sect. II. Art. 7. we had

$$P.z + P'.z' + P''.z'' + P'''.z''' + \&c. =$$
$$(P + P' + P'' + P''' + \&c.)\, Z.$$

giving

$$P.d.z + P'.d.z' + P''.d.z'' + P'''.d.z''' + \&c. =$$
$$(P + P' + P'' + P''' + \&c.)\, d.Z.$$

Wherefore, the equation $P.p + P'.p' + P''.p'' + \&c. = 0.$ is equivalent to

$$d.Z = 0.$$

Accordingly, when the weights equilibrate, this last equation subsists; and conversely.

When the common centre of gravity takes the highest or lowest position possible, we have

$$dz = 0.$$

and therefore, the system is in equilibrio. But the converse of this proposition, viz. that when $d.z = 0$, z is a maximum or minimum, though generally true, is liable to the usual exceptions for maxima or minima, discovered by making the differential equal to cipher. For if, as may happen in constrained motions, the common centre of gravity describes a curve of contrary flexure; at a point of that curve where the tangent is horizontal, the equilibrium shall subsist, although the common centre of gravity may not be then in the highest or lowest position it can take.

For the wedge, the power and resistances being applied in directions perpendicular to the back and faces, it was found that for equilibrium, the three forces should be proportional to the surfaces on which they act, *i.e.* to the three sides of the triangular end of the wedge, by which those surfaces may be represented.

To derive this theorem from the principle of virtual velocities, let the wedge be represented by (Fig. 130.) where B, L, L′ are the back and faces, α, β, γ, the angles to which they are opposed, and P, P′, P″ the forces acting perpendicularly on B, L, L′. The three forces being supposed to be directed to the same point, which may be taken for their common point of application, let this point describe the line ds, in the direction of the power P; the line of the motion shall then make, with the direction of the force P′, an angle equal to the angle γ, and with the direction of the force P″, an angle equal to the angle β; and we shall have

$$p : p' : p'' ::$$
$$1 : \cos.\gamma : \cos.\beta.$$

and, therefore, by the principle of virtual velocities,

$$P = P'.\cos.\gamma + P''.\cos.\beta.$$

This equation involves the three quantities P, P′, P″, and, therefore, does not express their relative magnitudes. To express this relation, the equation should contain but two of those quantities. And as the direction of *ds* is arbitrary, it may be supposed to be perpendicular to that of one of the forces; for then, the virtual velocity, estimated in the direction of that force, shall be cipher; and the corresponding term shall vanish from the equation. Accordingly, let *ds* be perpendicular to the direction of the force P″, *i. e.* parallel to the face L′; then

$$p'' = 0.$$

and therefore

$$P''.p'' = 0.$$

and the condition of equilibrium expressed by the principle of virtual velocities is

$$P.p + P'.p' = 0.$$

But the line *ds* makes, with the direction of the force P, an angle whose complement is β; and with the direction of the force P′ an angle whose complement is α. Wherefore,

$$p : p' :: \sin.\beta : \sin.\alpha :: L : B.$$

so that the proportion $P : P' :: p' : p$ is

$$P : P' :: B : L.$$

And in the same way, by taking *ds* in the direction of the face L, we get

$$P : P'' :: B : L'.$$

which two proportions may be stated as before,

$$P : P' : P''$$
$$:: B : L : L'.$$

Ordinarily, the triangle by which the wedge is represented, is isosceles, *i. e.* $L' = L$, and each of the angles at

the base is the complement of $\frac{\alpha}{2}$. in which case, the relation of the forces appears by the equation

$$\text{P} = \text{P}'.\cos.\gamma + \text{P}''.\cos.\beta.$$

which then becomes

$$\text{P} = 2\text{P}'.\sin.\frac{\alpha}{2}.$$

To ascertain the ratio of the power and weight for the inclined plane; let P, P′ be those forces, and let their common point of application be supposed to be carried up the plane through the space ds. Then, p, p' are as the cosines of the angles made by the plane with the directions of the forces P, P′. Wherefore, if e denote the angular elevation of the plane, and θ the angle made by the plane with the direction of the power, we shall have

$$p : p' :: \cos.\theta : \sin.e,$$

giving for equilibrium

$$\text{P}.\cos.\theta = \text{P}'.\sin.e.$$

If the power acts in a direction parallel to the plane,

$$\cos.\theta = 1. \qquad \text{and } \text{P} = \text{P}'.\sin.e.$$

7. To show what number of equations may be obtained from the general theorem, and in what manner they are to be applied, let θ, θ', θ'', &c. be the angles made by the directions of the forces P, P′, P″, &c. with the infinitesimal lines ds, ds', ds'', &c. described by their several points of application. The equation

$$\text{P}.p + \text{P}'.p' + \text{P}''.p'' + \&c. = 0.$$

is equivalent to

$$\text{P}.ds.\cos.\theta + \text{P}'.ds'.\cos.\theta' + \text{P}''.ds''.\cos.\theta'' + \&c. = 0. \qquad (a)$$

Now, referring the points of application to three rectangular axes, let the coordinates of the point of application of

the force P be x, y, z. Those of the point of application of the force P$'$, x', y', z', &c. Also, putting α, β, γ, for the angles made by the direction of the force P, with the axes of x, y, and z respectively; α', β', γ' for those made by the direction of the force P$'$ with the same axes, and so forth. Likewise, putting ϕ, χ, ψ for the angles made by the line ds, with the axes; ϕ', χ', ψ' for those made by ds' with the same, and so forth; we have, by a well known theorem,

$$\cos.\theta = \cos.\phi.\cos.\alpha + \cos.\chi.\cos.\beta + \cos.\psi.\cos.\gamma.$$

and

$$ds.\cos.\theta = ds(\cos.\phi.\cos.\alpha + \cos.\chi.\cos.\beta + \cos.\psi.\cos.\gamma) = dx.\cos.\alpha + dy.\cos.\beta + dz.\cos.\gamma.$$

And similarly,

$$ds'.\cos.\theta' = dx'.\cos.\alpha' + dy'.\cos.\beta' + dz'.\cos.\gamma'.$$

and

$$ds''.\cos.\theta'' = dx''.\cos.\alpha'' + dy''.\cos.\beta'' + dz''.\cos.\gamma''.$$

the coefficients of all the forces being expressed in like manner. Wherefore, making these substitutions, equation (a) becomes

$$\left.\begin{array}{l} \text{P}.\cos.\alpha.dx + \text{P}'.\cos.\alpha'.dx' + \text{P}''.\cos.\alpha''.dx'' + \text{P}'''. \\ \qquad \cos.\alpha'''.dx''' + \&c. \\ \text{P}.\cos.\beta.dy + \text{P}'.\cos.\beta'.dy' + \text{P}''.\cos.\beta''.dy'' + \text{P}'''. \\ \qquad \cos.\beta'''.dy''' + \&c. \\ \text{P}.\cos.\gamma.dz + \text{P}.\cos.\gamma'.dz' + \text{P}''.\cos.\gamma''.dz'' + \text{P}'''. \\ \qquad \cos.\gamma'''.dz''' + \&c. \end{array}\right\} = 0. \quad (b)$$

If the points of application are unconnected, so that the motion of one of those points shall have no influence upon another, then are the quantities p, p', p'', p''', &c. independent variables; and therefore, also $dx, dy, dz, dx', dy', dz'$, &c. are independent variables: whereby the last equation is resolvable into so many distinct equations, viz.

$$\text{P}.\cos.\alpha=0. \quad \text{P}'.\cos.\alpha'=0. \quad \text{P}''.\cos.\alpha''=0. \quad \&c.$$
$$\text{P}.\cos.\beta=0. \quad \text{P}'.\cos.\beta'=0. \quad \text{P}''.\cos.\beta''=0. \quad \&c.$$
$$\text{P}.\cos.\gamma=0. \quad \text{P}'.\cos.\gamma'=0. \quad \text{P}''.\cos.\gamma''=0. \quad \&c.$$

But if the points of application are connected in any manner, so that one of them cannot move unless by moving another, according to some certain law; then, the equations expressing the conditions to which the system of points is subjected, shall serve to eliminate so many variables from equation (b), and the coefficients of those that remain, being put equal to cipher, shall express the conditions to be satisfied for equilibrium.

8. If the forces are all applied to the same material point, we have

$$ds=ds'=ds''=ds''', \ \&c.$$

wherefore,

$$dx=dx'=dx''=dx''', \ \&c.$$
$$dy=dy'=dy''=dy''', \ \&c.$$
$$dz=dz'=dz''=dz''', \ \&c.$$

and equation (b) becomes

$$\left.\begin{array}{r}(\text{P}.\cos.\alpha+\text{P}'.\cos.\alpha'+\text{P}''.\cos.\alpha''+\&c.)\,dx\\ +(\text{P}.\cos.\beta+\text{P}'.\cos.\beta'+\text{P}''.\cos.\beta''+\&c.)\,dy\\ +(\text{P}.\cos.\gamma+\text{P}'.\cos.\gamma'+\text{P}''.\cos.\gamma''+\&c.)\,dz\end{array}\right\}=0. \qquad (c)$$

which, as dx, dy, dz, are independent variables, is reducible to three distinct equations, agreeably to Sect. I. Art. 11.

If the point to which the forces are applied, is confined to a surface whose equation is $\text{L}=0$. the differentials dx, dy, dz, are no longer absolutely independent of each other. Wherefore, equation (c) is no longer resolvable into three. But putting this equation in the form $\text{X}.dx+\text{Y}.dy+\text{Z}.dz=0$.

and eliminating dx between this and the differential equation of the surface, viz.

$$\frac{d\text{L}}{dx}.\,dx+\frac{d\text{L}}{dy}.\,dy+\frac{d\text{L}}{dz}.\,dz=0.$$

we have

$$\text{Y}.\,\frac{d\text{L}}{dx}.\,dy+\text{Z}.\frac{d\text{L}}{dx}.\,dz-\text{X}.\frac{d\text{L}}{dy}.\,dy-\text{X}.\frac{d\text{L}}{dz}.\,dz=0.$$

In this equation, the variables dy, dz, are independent: wherefore, their coefficients are, each, equal to cipher, giving for the conditions of equilibrium

$$\text{Y}.\frac{d\text{L}}{dx}-\text{X}.\frac{d\text{L}}{dy}=0. \qquad \text{Z}.\frac{d\text{L}}{dx}-\text{X}.\frac{d\text{L}}{dz}=0.$$

For example, let the equation of the surface be $a.dx+b.dy+c.dz=0$. we have

$$\frac{d\text{L}}{dx}=a. \qquad \frac{d\text{L}}{dy}=b. \qquad \frac{d\text{L}}{dz}=c.$$

which values being substituted in the general formula, we have the equations

$$\text{Y}.a-\text{X}.b=0 \qquad \text{Z}.a-\text{X}.c=0.$$

The elimination is more expeditiously performed by adding the differential equation of the surface, multiplied by an indeterminate coefficient, to the general equation of equilibrium: then putting the coefficient of each variable equal to cipher, and eliminating the indeterminate coefficient. Thus,

$$\text{X}+m.\frac{d\text{L}}{dx}=0. \qquad \text{Y}+m.\frac{d\text{L}}{dy}=0. \qquad \text{Z}+m.\frac{d\text{L}}{dz}=0.$$

from which, eliminating m, we have

$$\text{Y}.\frac{d\text{L}}{dx}-\text{X}.\frac{d\text{L}}{dy}=0. \qquad \text{Z}.\frac{d\text{L}}{dx}-\text{X}.\frac{d\text{L}}{dz}=0.$$

the same with those obtained by directly eliminating dx, be-

tween the general equation $X.dx+Y.dy+Z.dz=0$. and the equation of the surface.

9. Equations (c) will serve to show the resultant of any number of forces applied to a material point: for any one of the equilibrating forces, its direction being changed into the opposite, is the resultant of the remaining forces.

Thus, to find the resultant of two forces, there are but three terms in each of those equations. And the plane of the forces P, P', being taken for that of x, y, we have

$$\cos.\gamma=0. \qquad \cos.\gamma'=0.$$

and therefore,

$$\cos.\gamma''=0.$$

Wherefore, the equations by which the resultant of the forces P, P', is to be determined in quantity and direction, are

$$P''.\cos.\alpha''=P.\cos.\alpha+P'.\cos.\alpha'.$$

$$P''.\cos.\beta''=P.\cos.\beta+P'.\cos.\beta'.$$

The forces P, P', P'', being represented in quantity and direction by three right lines drawn from their common point of application, the terms of the equations express the coordinates of the extreme points of those lines: and the equations show that each of the coordinates of the extremity of the line P'', is equal to the sum of the corresponding coordinates of the extremities of the lines P, P', *i. e.* that the coordinates of the extremity of the line P'' are those of the extremity of the diagonal of the parallelogram whose sides are the lines P, P', and, therefore, that the line P'' is that diagonal.

In the same way, and without the trouble of elimination, it may be seen that the resultant of three forces applied to a material point, is represented in quantity and direction by the diagonal of the parallelopiped, whose three sides represent the three component forces in quantity and direction.

10. To find the conditions of equilibrium for forces applied to a rigid system, it is to be observed, that the system cannot be moved, except by a movement of translation, by which all the points describe equal and parallel lines; or, by a motion of rotation. For the first kind of movement we have

$$dx = dx' = dx'', \text{ \&c.}$$
$$dy = dy' = dy'', \text{ \&c.}$$
$$dz = dz' = dz'', \text{ \&c.}$$

which changes the equation (b) into

$$\left.\begin{aligned} &(\text{P}.\cos.\alpha + \text{P}'.\cos.\alpha' + \text{P}''.\cos.\alpha'' + \text{\&c.})\, dx \\ +&(\text{P}.\cos.\beta + \text{P}'.\cos.\beta' + \text{P}''.\cos.\beta'' + \text{\&c.})\, dy \\ +&(\text{P}.\cos.\gamma + \text{P}'.\cos.\gamma' + \text{P}''.\cos.\gamma'' + \text{\&c.})\, dz \end{aligned}\right\} = 0.$$

giving

$$\text{P}.\cos.\alpha + \text{P}'.\cos.\alpha' + \text{P}''.\cos.\alpha'' + \text{\&c.} = 0.$$
$$\text{P}.\cos.\beta + \text{P}'.\cos.\beta' + \text{P}''.\cos.\beta + \text{\&c.} = 0.$$
$$\text{P}.\cos.\gamma + \text{P}'.\cos.\gamma' + \text{P}''.\cos.\gamma'' + \text{\&c.} = 0.$$

which are the three first conditions of equilibrium among a number of forces, applied to the points of a rigid system.

Any rotatory motion which may be given to the system, is resolvable into motions round the axes of the coordinates. Now if the angle $d\theta$ is described round the axis of z, no change is made thereby in the magnitude of any of the lines z, z', z'', &c. which are the coordinates of the points of application parallel to that axis, therefore

$$dz = dz' = dz'', \text{ \&c.} = 0.$$

Moreover,

$$d\theta = \frac{dx}{y} = -\frac{dy}{x} = \frac{dx'}{y'} = -\frac{dy'}{x'}, \text{ \&c.}$$

or,

$$dx = y.d\theta, \quad d.x' = y'.d\theta, \text{ \&c.}$$
$$dy = -x.d.\theta, \quad dy' = -x'.d\theta, \text{ \&c.}$$

and these values being substituted in equation (b), it becomes

$$\text{P}\,(y.\cos.\alpha - x.\cos.\beta) + \text{P}'\,(y'.\cos.\alpha' - x'.\cos.\beta') + \&\text{c.} = 0.$$

In the same manner, for a movement of rotation round the axis of y, we shall have

$$\text{P}\,(x.\cos.\gamma - z.\cos.\alpha) + \text{P}'\,(x'.\cos.\gamma' - z'.\cos.\alpha') + \&\text{c.} = 0.$$

And for a movement of rotation round the axis of x, the equation

$$\text{P}\,(z.\cos.\beta - y.\cos.\gamma) + \text{P}'.\,(z'.\cos.\beta' - y'.\cos.\gamma') + \&\text{c.} = 0.$$

which completes the six conditions of equilibrium for a system of invariable form, agreeably to Sect. IV. Art. 1.

DYNAMICS.

SECTION I.

THE MEASURES OF MOTION AND FORCE.

1. That branch of mechanical science which relates to unbalanced forces and their effects, is called Dynamics: and as the effect of unbalanced force is motion, it seems expedient, in the first place, to treat of motion, independently of the causes by which it is produced.

When a body changes its place, it is said to be in motion; and the rate of that change is called its velocity. Now velocity is not itself a mathematical quantity; but if it can be measured by quantities of this nature, it thereby falls within the province of mathematical science.

The velocity of a body is greater or lesser, according as the space described by it in a given time is greater or lesser. Wherefore, the space described in a given time is the measure of the velocity: and as in comparing the velocities of different bodies, or of the same body at different periods, by the spaces described, the times must be equal, it becomes necessary in every case of such comparison, to fix on some portion of time for this purpose. This portion is denominated the unit of time. The velocity is, therefore, said to be measured by the space described in the unit of time.

Whatever space a body describes in a unit of time, it is evident that with the same velocity, it would describe twice

that space in two such units; and generally, that the space described with the same velocity, is obtained by multiplying the space described in a unit of time, by the number of such units contained in the time of the motion. Wherefore, v, denoting the velocity; t, the number of units in the time of the motion; and s, the space described, there is $s = vt$. or $v = \frac{s}{t}$. of which it is to be remarked, that whatever be the magnitude of s, it becomes, when divided by t, the space described in a unit of time.

In all cases, such as this, wherein quantities of different kinds seem to be compared together, the terms are rendered homogeneous by regarding all, or all but one, as numbers. Thus in the preceding equation not only v and t, but s also may be regarded as a number; viz. the number of units of space contained in the space described. These units of measure for the several quantities which are involved in the same equation are, all but one, purely conventional. For example, if 1″ is taken for the unit of time, and one foot for the unit of space, then the unit of velocity is that of a body which describes one foot in a second of time. So that whatever be the space described, or the time of describing it, the quantity $\frac{s}{t}$ shall be the number of feet described in a second, or, which is the same thing, the number of units in the velocity of the moving body. In these estimates, the velocity is supposed to be uniform; *i. e.* it is supposed that the body describes equal spaces in equal times during the whole continuance of the motion: and the same thing is always to be understood, when not otherwise expressed.

For the more ready solution of questions relative to such motions, it will be convenient to compute the time from the moment of the passage through some one point; and also to denote the place of the body, at any instant, by its distance from some point taken at pleasure in the line of its motion.

Thus, let the body move in the direction AB (Fig. 131.) with the velocity v, and let the time be measured from the moment of its passage through D; *i. e.* let D be the point for which $t = o$. Then if m is the place of the body after any time t, there shall be $\text{D}m = vt$. Futher, let C be the point from which the distance of the body is to be measured. Putting s for this variable distance, and a for the line CD, we shall have $s - a\,(= \text{D}m) = vt$. and $s = a + vt$. If the point C is taken at the other side of D, it is evident that the sign of a must be changed.

If a second body moves uniformly in the same line with the velocity v', let D′ be the point through which it passes at the instant from which the time is computed, *i. e.* at the same instant in which the former body passes through D; and let s' be its distance from C, and a' the distance CD′; there will be in like manner for this body, $s' = a' + v't$. Now if it is required to find the time of the meeting of these bodies, this is found by making $s = s'$ in the two last equations, which gives $t = \frac{a' - a}{v - v'}$. and by substituting this value of t, in the expressions for s or s', there is $s = \frac{a'v - av'}{v - v'}$. When the value of t, given by the former of these two equations, is negative, it is inferred that the meeting has occurred before the instant, from which the time is computed, *i. e.* before the arrival of the one body at D, or that of the other at D′. and when the denominator of that value is cipher, *i. e.* when $v = v'$, the value itself is infinite: which imports that the bodies never meet. This indeed is evident of itself; inasmuch as the bodies, in that case, move in the same direction with the same velocity.

2. If the spaces successively described in equal times constitute an increasing series, the motion of the body is said to be accelerated; and if they constitute a decreasing series, the motion is retarded. When this occurs for portions of

time, however minute, the velocity is continually varied; and in such cases, it may not be so obvious how the velocity of the body, at any one instant of time, or point of space, is to be measured. For if the velocity is accelerated, the space described in any portion of time, is greater than that which would be described with the velocity at the beginning, and less than that which would be described with the velocity at the end of that portion of time; and the contrary if the velocity is retarded; *i. e.* v and v' denoting these velocities, there is $v < \frac{s}{t} < v'$, for one case, and $v > \frac{s}{t} > v'$, for the other. This difficulty is met by supposing s, and consequently t, to be indefinitely diminished: for the change of velocity, however rapid, is gradual. Whence it is plain, that by diminishing the space, and consequently the time, indefinitely, v and v', the velocities at the extreme points, shall approach within any assignable difference; and therefore, that the quantity $\frac{s}{t}$, which is ever intermediate between them, shall be ultimately equal to either. Accordingly, ds and dt, denoting the indefinitely small increments of space and time, there must be,

$$v = \frac{ds}{dt}.$$

These things being established, it may be satisfactory to see how the space, time, and velocity, as dependent, each of them, on the other two, may be graphically represented.

3. Let a right line such as AB and its parts, (Fig. 132.) represent the time of the motion and the parts of that time, and let a continued line $bb'b''b'''$, &c. be drawn, such that its perpendicular ordinates shall be proportional to the velocities at the moments of time corresponding to the points of the line AB on which they are raised: the spaces described by the body in any portions of the time of its motion, shall be proportional to the areas standing on the cor-

responding portions of the base. For if the time AB is divided into the equal portions aa', $a'a''$, $a''a'''$, &c. and if the velocity continued unvaried during each of those portions of time, the spaces described in those times would be represented by the rectangles am, $a'm'$, $a''m''$, &c. or by the rectangles ab', $a'b''$, $a''b'''$, &c. according as it is the velocity at the beginning or ending of each portion of time that is supposed to continue unvaried during that time: and therefore, the space described in the whole or any portion of the time AB, would be represented in magnitude by the area of the inscribed or circumscribed serrated figure standing on the whole or that portion of the line AB. This will hold however frequently the velocity is supposed to change, *i. e.* however minutely the line AB is divided. It therefore holds for the limit of the space so described by the body, and for that of the serrated figure by whose area it is represented. But the limit of the former is the space described by the body moving with a velocity continually varied; and that of the latter is the area of the figure wherein the serrated boundary is replaced by the continued line $bb'b''$, &c. For, by producing each of the right lines bm, $b'm'$, $b''m''$, &c. until they intersect the greatest ordinate, it will appear that the excess of the area of the circumscribed above that of the inscribed figure, is the difference between the greatest and least of the elementary rectangles: *i. e.* putting l, and l' for the greatest and least ordinates, and m, for the base of each of the elementary rectangles, the difference is $m(l-l')$ which vanishes with m. And since, by the continual subdivision of the line AB, the circumscribed and inscribed figures shall be made to approach within any assignable difference, the same must hold *a fortiori* when either of them is compared to the figure bounded by the continued line $bb'b'$, &c. which is intermediate between them.

4. Having shown how motions are measured, we may proceed to show how they are compounded and resolved.

Perhaps there is no motion observed in nature that is not the resultant of several other motions. Thus if a body is moved in the hold of a ship, whilst the ship moves on the surface of the sea, and the surface of the globe revolves from west to east, round its axis, and the earth itself proceeds in its annual course round the sun, and the sun, together with the whole planetary system, moves round some of the nearer constellations, the motion of the body in absolute space is compounded of all these motions.

It is now to be shown how the resultant motion is to be collected from its components, and *vice versa;* and to begin with the simplest case, that of the composition of two motions, it may be stated, that if the two component motions are represented in quantity and direction by two sides of a parallelogram, the resultant shall be represented in quantity and direction by the diagonal of that parallelogram. This theorem has been already domonstrated in the note appended to Statics, Sect. I. Art. 2. in addition to which the following may serve as an exemplification. If a body moves from A to B in the right line AB (Fig. 133.) whilst that line, supposed rigid, is carried to the position A″B″; at the end of that time the body shall be found, not at B, but at B″, and if the velocity of the body in the rigid line is always to that of the line itself, in the ratio of AB to AA″, the line being always parallel to AB, which is its first position, the absolute motion of the body shall be in the right line connecting the points A and B″. For the velocities being constantly in the ratio of AB to AA″, the lines described by these motions in the same time, shall be in the same ratio: and these lines being the coordinates which mark the position of the body at any one instant, it follows, that the body is always to be found in the right line AB″.

5. What has been shown respecting the composition of two motions in different directions may be extended to the composition of any number of coexisting motions in any di-

rections, by continually compounding a new motion with the partial resultant already discovered. This method of successive compositions is easily practised. Thus, if a body moves through a certain space in one direction, and a certain space in another direction during the same time; it has been shown that, at the end of that time, it shall be found at the extremity of the diagonal of the parallelogram whose sides are the lines it would have described by the separate motions. Wherefore, the place of the body at the end of the given time is found, by drawing the line it would have described by one of those motions singly, and from its extremity a line equal and parallel to that which it would have described by the other motion singly; *i. e.* it will be found at the same point, at which it would have arrived had these motions been successive instead of cotemporaneous. For example, if AB is the line it would have described by one of these motions, (Fig. 134.) and AC that which it would have described by the other in the same time; the point at which it arrives, by the compound motion, is found by supposing it to have been carried by the former motion to B, and to be subsequently carried over the line BC′ equal and parallel to AC. Now if a third cotemporaneous motion had been communicated to the body, by which alone it would have been carried over the line AD in the same time, this may be supposed not to have commenced, until after its arrival at C′; and by drawing from this point a line, C′D′, equal and parallel to AD, we have the point D′, to which it arrives, in consequence of the three motions simultaneously communicated; and if each of these motions were uniform and rectilinear, the right line AD′ shall be that actually described by the compound motion; and this motion shall be also uniform, bearing to any of the partial motions, the ratio of the line AD′ to that by which the partial motion is represented. From all which it is inferred, that if several motions are simultaneously communicated, the point of space at which the body is found at

the end of any portion of time, is the same at which it would arrive, if these movements were communicated successively. Accordingly, the place of the body to which these movements are simultaneously communicated, and the line described by it in any given time, are found as follows. From the body let the several lines be drawn which would have been described by it in a given time, for example, 1″, by the separate motions: then from the extremity of one of those lines, let a line be drawn equal and parallel to a second; and from the extremity of this, another equal and parallel to a third; and so on, until all the lines originally drawn from the body, except the first, are thus transferred: the extremity of the last line is the point where the body shall be found at the end of 1″, when the several movements are coexistent; and the line actually described, shall be that connecting the points occupied by the body at the beginning and end of the motion. If the line last drawn terminates in the point of departure, the polygon is closed, and the absolute motion of the body is nothing.

It cannot escape notice, that all which has been here shown, relative to the composition of motions, is strictly coincident with that before established, respecting the composition of forces, Sect. 1. Art. 8.

This composition of motions may be easily imagined by presenting different examples. Thus, if a man climbs to the topmast of a ship, whilst the ship itself is in motion, his actual motion is compounded of the motion of the vessel, and of that which is the consequence of his own exertions. Thus, also, if a body moves on a flat surface, whilst this surface is moved on another surface, and the second on a third, and so forth, for any number of surfaces, the motion of the body shall be compounded of all these several motions, and the point at which it shall be found at the end of any given time, for example, 1″, shall be that at which it would have arrived, if it had been directed by these motions successively, each of

them being continued for 1″. The courses are indeed different, as are also the times. In the latter case, the figure is a polygon, and each of its sides is described in 1″; and in the former case, the body describes in 1″ the line which closes the polygon: but the point to which it arrives is the same, whether the motions are simultaneous or consecutive.

6. As motions are compounded, so are they also resolved in the same manner as forces: the resolved motions being represented, in quantity and direction, by the sides of a triangle or polygon constructed on the line representing the motion to be resolved. But the relation between the resultant motion and its components is exhibited in the way best fitted for analysis, by resolving the motion according to three rectangular axes: and this also is done in the same way as for forces. The motion in each axis is had by multiplying the line representing that motion into the cosine of the angle which its direction makes with that axis: and all the movements, whether simultaneous or consecutive, being treated in this way, the problems respecting curvilinear motion may be reduced to those relative to rectilinear motion; and the whole theory of dynamics unfolded, not only with greater facility, but also in a manner more fertile in general results.

7. The composition and resolution of motion being so far explained, the subject which next offers itself for consideration is the cause of motion: and for this we are compelled to look beyond the body moved. For a body at rest is observed to continue in the same condition, until it is disturbed by something external to itself; and as it cannot produce motion in itself, so neither can it increase the velocity of that already imparted to it. But whether a body set in motion has any natural tendency to rest, whereby it is disposed to return to that state, is a question not so readily answered. For though, like the former question, relative to the generation of motion, this is to be decided by an appeal

to facts, yet the facts are such as require a more attentive consideration. It is observed of all motions near the earth's surface, that they require for their continuance a continued renewal of the impressions by which they had been produced; without which, if not suddenly checked, they all gradually decay, until at length the bodies return to a state of rest. Thus, if a ball is rolled on a level plane, its motion is continually retarded; and after describing a space greater or less, the motion is altogether lost. And in the same way a pendulum, being let fall from a certain height, oscillates through arches continually decreasing, until at length it becomes quiescent at the lowest point of the arch.

From this seemingly spontaneous relaxation of motion, it might be supposed that there is in matter a natural tendency to rest, or a preference of rest to motion. But before that this conclusion can be drawn, it should be considered how far the observed retardations may arise from external impediments: The existence of such impediments is certain: the attrition of the surface on which the body moves is one, and the resistance of the medium in which it moves is another. These obstructions cannot be altogether removed, but they may be reduced, and then it is invariably observed, that the retardations are reduced in the same proportion, and the motion is continued for a period suitably longer. Thus, by levigating the plane, the body which rolls on it proceeds to a greater distance. By polishing the axle on which the pendulum oscillates, in order to diminish the attrition, the number of vibrations performed is suitably increased; and the reduction of the diameter of the axle, *i. e.* the diminution of the leverage by which the force of friction acts against the moving body, has the same consequence. The retardation of motion in a resisting medium is less, as the density of the medium is less; and in the same medium it is diminished, by increasing the weight of the moving body, in relation to the surface which it

opposes to the reaction of the medium. By such contrivances, the resistance opposed to the motions of bodies can be greatly diminished, and the result is such as to warrant the inference, that if all impediment could be completely removed, there would be no loss of motion sustained. In this way we are obliged to reason respecting motions near the earth's surface; where, though we may diminish, we cannot completely remove all manner of obstructions. But the proof afforded by the heavenly bodies is decisive of the question; the motions of these bodies having continued for ages, without the least sensible abatement.

The perfect inactivity of matter being then admitted, or its incapacity to make any change in its own condition, as to rest or motion, it follows, that every such change, whether in the quantity or direction of the motion, is to be regarded as an indication of the action of some external force; which leads us to consider how the force exerted is measured by the motion or change of motion produced by it. The question will occur, whether is it to be measured, simply, by the quantity of motion, or by its square, cube, or what other function? And whether by the effect produced by its action, through a given space, or during a given time? This doubt as to the measure of force by motion, is easily removed by the observation, that the motion or change of motion produced in a body, by the action of a given force during a given time, is the same, whether the body subjected to this action is at rest or in motion; and if in motion, whatever be the direction of the force applied with respect to that of the motion already existing. Thus, all bodies on the earth's surface move from west to east round the earth's axis, and are carried in the same direction by the progressive movement of the earth round the sun: yet, if a body partaking of such a motion, is acted on by a certain force during a certain time, the same effect is produced, whether

the force is directed from east to west, or from west to east, or in any transverse direction whatsoever. The same observation may be extended to a body contained in a ship in motion. The relative velocity produced by a force is the same, in whatever direction the force is applied.

Now, the change of motion produced in a given body, by a given force, acting during a given time, being independent of the state of the body to which the force is applied, it follows, that a double or triple force must produce a double or triple change of motion; and, generally, that the force is proportional to the change of motion produced by it in a given time; the change of motion being estimated in the direction of the force. For whatever be the magnitude of the force, if it is divided into a number of units, the change of motion produced by each unit of force is the same, whether the remainder of the force has acted or not; and, therefore, the total change of motion is the aggregate of the motions, which the several units of force would have produced, if separately applied to the body at rest, *i. e.* the motion, or change of motion, produced in a given body, during a given time, is proportional to the force applied; and is, therefore, the adequate measure of that force.

From observations of this nature, to which there is no exception, the two following laws of motion are collected:

1st, Every body continues in its state of rest, or of uniform rectilinear motion, unless influenced by some external force.

2nd, Every motion or change of motion produced in a given body, during a given time, is proportional to the exciting force, and in the same direction.

The first of these laws is called the law of Inertia; the second exhibits the measure of force, or the relation which subsists between the force and the velocity: and the whole theory of Dynamics is but the developement of these two

propositions. Thus, from the second law of motion it follows, that if two forces are simultaneously applied to a body, each of which is fitted to move it over a certain right line in a given time, these lines being made the sides of a parallelogram, its diagonal shall be the space described by their united actions. For the motion excited by each force, being uninfluenced by the effect produced by the other, the motions produced by each force acting separately, must subsist together; and, therefore, the motion of the body must be that which arises from the composition of those motions. It follows also, that if a force is applied to a body in motion, and oblique to its direction, the effect shall be the same as if it were applied to the body at rest, in conjunction with another force fitted to produce the previous motion.

In the second law of motion the force was measured by its effect produced by its action during a given time, and on a given mass, and it is therefore requisite to show how forces are to be compared when acting during different times or on different masses. Now, it follows, from what has been already established, that the motion, or change of motion, produced in a given body by a given force, is proportional to the duration of its action. For the time being divided into any number of equal portions or units, the change produced in any unit after the first, is uninfluenced by the motion already produced: and the changes of motion in all these equal portions of time, being equal and in the same direction, the total change of motion must be equal to the change produced in one of these moments multiplied by their number; therefore, t representing the number of units in the whole time, v the velocity produced in that time, and f the force, measured by the velocity produced by it in the unit of time, we have $v = ft$, or $f = \frac{v}{t}$.

To show how forces are to be compared by the effects

produced on unequal masses of matter, it may be laid down as a principle, that when two equal forces act on two equal and contiguous masses of matter, during the same time, and in parallel directions, there is no reason why a greater degree of velocity should be produced in one of the masses, rather than in the other, all influential circumstances being the same. The masses, therefore, shall move with the same velocity, and in parallel directions; and there being no relative motion of one from the other, they shall remain contiguous, both during the action of the forces, and after they shall have ceased to act. The bodies, therefore, may be supposed to be connected together, so as to constitute one mass equal to their sum; and this, without inducing any change in the velocity produced. Accordingly, whatever force is required by its action during a given time, to produce a certain velocity in a certain mass, a double force will be required to produce, in the same time, the same velocity in a double mass; a triple force, to produce the same velocity in a triple mass, &c. And generally, if m denotes the number of units of mass in the body considered, and if f is the force which in the time t, produces the velocity v, in the unit of mass; then the force F producing in the same time the same velocity in the mass m, shall be mf. *i. e.* $\mathrm{F}=mf$. But it has been already shown, that $f=\frac{v}{t}$. $\therefore \mathrm{F}=\frac{mv}{t}$. wherein the symbols m, v, t, denote numbers.

In proceeding to connect the various phenomena of motion with the forces, which are their efficient causes, we shall begin by supposing the forces applied to a material point: or if, in treating of this part of the subject, bodies are spoken of, all the equal molecules of the body shall be supposed to be influenced by equal forces, applied to them in parallel directions. Therefore, in this first volume, where the subject is thus restricted, our inquiries shall be confined to the consideration of motions merely progressive. When the con-

nexion between the phenomena and their efficient causes is so far understood, the student will be prepared to enter on the subject of rotatory motion, or the effect of force, when unequally applied to the equal portions of the same mass; which shall be treated of in the second volume of this work.

SECTION II.

OF FORCES PRODUCING RECTILINEAR MOTION.

1. WHEN a material point is disturbed from a state of rest by the action of a constant force, the increments of velocity generated in equal times are equal, *i. e.* the velocity of the moving body is one uniformly accelerated. The velocities thus generated, are as the number of those equal increments, *i. e.* as the number of equal portions of time counted from the beginning of the motion; wherefore they are proportional to the times in which they are acquired. Accordingly, the accelerating force f, being estimated by the velocity generated by it in a unit of time, the velocity v, generated in any time t, shall be expressed by the equation

$$v = f.t. \tag{1}$$

To show the dependance of the space on the time and velocity, it will be convenient to make use of the theorem proved in Art. 3. of last section; and to accommodate that theorem to the present case, wherein the generated velocities are as the times counted from the beginning of the motion, the ordinates must be taken proportional to the abscissæ. Accordingly, if in (Fig. 133.) the point A corresponds to the instant at which the velocity is cipher, and if the velocity acquired in the time AB, is represented by the ordinate BC, that which is generated in the time AM, shall be represented by an ordinate MN, which is to BC in the ratio of AM to AB. When such is the law of the ordinates, the line by which they are terminated shall be a right line diverging from AB, and the scheme a right-lined triangle. The areas of the similar triangles AMN, ABC, being as the squares of the bases

AM, AB, or of the altitudes MN, BC, it follows, that the spaces described with a velocity uniformly accelerated, and counted from the beginning of the motion, are as the squares of the times in which they are described, or as the squares of the velocities acquired in those times. For example, the spaces described in 1″, 2″, 3″, 4″, &c., counted from the beginning of the motion, are as the numbers 1, 4, 9, 16, &c. And the spaces described in the several equal moments are as the odd numbers 1, 3, 5, 7, &c., these being the differences of 0, 1, 4, 9, 16, &c. the squares of the natural numbers, including cipher.

If the velocity is uniformly retarded, *i. e.* if it suffers equal decrements in equal times, the velocities are proportional to the times, counted to the end of the motion; and the ordinates, by which they are represented, are terminated by a right line converging to the base line, by which the time is represented. Therefore, the scheme given in (Fig. 133.) will serve to represent the theory of uniformly retarded velocities, by regarding A as the point of time at which the motion terminates. Thus, the spaces described in the times BA, MA, being proportional to the areas BAC, MAN, are as the squares of the times BA, MA, or of the initial velocities BC, MN.

If a body moves with a velocity uniformly accelerated, the space described, measured from the beginning of the motion, shall be one-half of that which it would describe with the last acquired velocity continued uniformly for an equal time: the former of these spaces being represented by a triangle, and the latter by a parallelogram of the same base and altitude.

The following theorems are now established relative to the effects of a constant force, on a body moved by it from a state of rest.

1st, The velocities generated in any times, counted from the beginning of the motion, are proportional to those times.

2d, The spaces described, measured from the beginning of the motion, are proportional to the squares of the times, or of the acquired velocities.

3d, The space described from the beginning of the motion, is one-half of that which would be described with the acquired velocity, continued uniform for an equal time.

The same theorems are equally applicable, when the velocity is uniformly retarded; the spaces and times being measured to the end of the motion.

The first of these theorems is expressed by the equation (1), wherein f denotes the force, or its measure, which is the velocity generated in a unit of time; and t, as before, the number of such units in the time of the motion.

The third theorem is expressed by the equation $2s = vt$. Whose terms being multiplied into those of equation (1), there is

$$s = \frac{ft^2}{2}. \tag{2}$$

whereby the space is known when the time is given; and conversely. By multiplying the alternate members of the same equations, there is

$$v^2 = 2fs. \tag{3}$$

whereby the velocity is known when the space is given, and conversely.

If the body has an initial velocity v', that with which it moves at the expiration of any time, t, is expressed by the equation

$$v = v' \pm ft. \tag{4}$$

The force f, being taken with an affirmative or negative sign, according as the velocity is accelerated or retarded. The velocity, expressed by this equation, is the sum or difference of two velocities, of which one is the uniform velocity v', and the other the velocity ft, proportional to the time t.

The space described is the sum or difference of the spaces which would have been severally described in these two ways. That due to the uniform velocity v' is had by multiplying this velocity into the time; and that due to the uniformly accelerated velocity, by multiplying ft, the velocity generated in the time t, by half that time. The former part is, therefore, $v't$, and the latter $\frac{ft^2}{2}$. which gives for the space, the equation

$$s = v't \pm \frac{ft^2}{2}. \qquad (5)$$

Exterminating t, between equations (4) and (5), there is

$$2fs = (v^2 - v'^2). \qquad (6)$$

Wherein f is to be taken with a negative sign, when the motion is retarded.

2. As near the surface of the earth, bodies descending or ascending in the vertical, are known to be uniformly accelerated or retarded, we have only to put for f, its proper value in the foregoing formulæ, in order to have the answer to every inquiry, whether relating to the space of ascent or descent, the velocity lost or gained, or the time required for this purpose. Now, it is ascertained by experiments made with pendulums, that in the latitude of 45°, a body, falling freely, would describe sixteen feet and one inch in a second of time, wherefore, the velocity acquired in the same time, is that which would carry the body through 32 feet 2 inches in one second. In this case, the force is that of gravity at the earth's surface, which it is convenient to designate by an appropriate symbol. Thus f, being replaced by g, in the preceding equation, they will be known to relate to bodies ascending or descending, under the influence of the force of terrestrial gravity.

If a body is projected vertically upwards with a certain velocity v', the diminished velocity v, corresponding to any time, or the time corresponding to that velocity, is given

by equation (4), which as the value of g is negative, becomes $v = v' - gt.$ or $v' - v = gt.$ The same observations apply when equations (5) and (6) are used to compare the time or velocity with the height of ascent.

By making $v = 0.$ in equation (6), there is for the whole height of ascent, $s = \frac{v'^2}{2g}.$ which compared with equation (3) or $s = \frac{v^2}{2g}.$ shews, that if the spaces are equal, the velocity destroyed in one body is equal to that generated in the other: or that the body, on its return to the surface of the earth, regains precisely the same velocity with which it had been projected upwards; and it also shows, that if a body is returned upward with the velocity acquired in the descent, it shall rise to the same height from which it fell. Also, by comparing equations (1) and (4), it will be seen that in the same case the times of ascent and descent are equal. Finally, from same equations it will follow, that if one body were to begin to ascend, and the other to descend at the same instant, the sum of the velocities in the ascending and descending bodies, shall be constantly equal to that with which the former had been projected upwards. For equation (4) gives for the velocity of the ascending body, $v = v' - gt.$ and equation (1) for the descending body, $v = gt.$ and the sum of these values is v'; which is also the relative velocity of one of the bodies from or towards the other.

Let the numerical value of g be $32\frac{1}{2}$, viz. the number of feet through which a body would be carried in a second of time with the velocity generated by gravity in that time; and let this value be assigned to f or g, in those equations; then, t shall denote the number of seconds in the time of the motion; s, the number of feet in the space described; and v shall express the number of feet which a body moving uni-

formly with that velocity would describe in one second. Accordingly, the equation between the time and velocity is

$$v = 32t'' \qquad (a)$$

omitting the two inches in the value of g, for sake of simplicity.

The equation between the time and space is

$$s = 16t''^2. \qquad \text{or, } t'' = \frac{\sqrt{s}}{4}. \qquad (b)$$

And that between the space and velocity is

$$v = 8\sqrt{s}. \qquad \text{or, } s = \frac{v^2}{64}. \qquad (c)$$

Thus, if the body falls through a space of 100 feet, by equation (b) there is $t'' = \frac{10''}{4} = 2\frac{1}{2}''$. and by equation ($c$) $v = 80$ feet per $1''$.

3. When a body is in part sustained by an inclined plane, the force by which it is solicited is had by resolving g, the force of gravity, whose direction is vertical, into two forces; one of them perpendicular to the plane, and the other in the direction of the plane. These are $\frac{b}{l}.g$, $\frac{h}{l}.g$, in which, l, h, b denote the length, height, and base of the plane. The force $\frac{b}{l}.g$, is destroyed by the resistance of the plane, but the force $\frac{h}{l}.g$, produces its full effect. And as this value is the same in every part of the plane, the motion of the body descending by this force is uniformly accelerated. Accordingly, all questions relative to the motion of such a body, are to be solved by making $f = \frac{h}{l}.g$, in equations (1), (2), (3). which then are

$$v = \frac{h}{l}.g.t. \qquad (d)$$

$$s = \frac{h}{2l}.g.t^2. \qquad (e)$$

$$v^2 = 2\frac{h}{l}.g.s \qquad (f)$$

Equation (d) shews that the velocity acquired in any time, by a body descending down an inclined plane, is to that acquired in the same time, by a body descending freely in the vertical, as the height of the plane to its length.

The space described in a given time, or the time of descent through a given space, is had from equation (e). Thus, putting l for s in that equation, we have for the time of descent down the entire plane,

$$t = l.\sqrt{\frac{2}{gh}}.$$

From which it appears, that the times of descent down different planes, are proportional to their lengths, divided by the square roots of their heights: and that if the heights are equal, the times are simply as the lengths of the planes. The height itself being included among those planes, it follows, that the time of descent through the length of the plane, is to the time of descending through its vertical height as the length to the height. This appears also by a comparison of the values of those times, viz. $l.\sqrt{\frac{2}{gh}}$, and $\sqrt{\frac{2h}{g}}$, which are in the proportion of l to h.

If it is required to ascertain the portion of the plane through which a body would descend, whilst another body falls down the vertical height, we have by the same equation,

$$t^2 = \frac{2l.s}{gh}.$$

and putting this equal to $\frac{2h}{g}$, which by equation (2), ex-

presses the square of the time of the fall down the vertical height, we have

$$ls = h^2, \qquad \text{or,} \ s = \frac{h^2}{l}.$$

showing that the portion of the inclined plane described in the time of the fall down the vertical height, is a third proportional to the length and height. Accordingly, if AB is the inclined plane, and BC its height, (Fig. 135.) and CD a perpendicular on the plane from the point C, the segment BD shall be the portion required. For by similar triangles, AB : BC : BD.

If a circle is described round the diameter BC, the several chords drawn from B, such as BD, BE, BF, &c. shall be described by bodies descending along them in the same time; the time of descent for each being equal to that of the descent down the vertical diameter BC. And the same thing is true of the chords DC, EC, FC, terminated by the other extremity of the vertical diameter; which is plain, by considering that these are the same as to length and inclination, with the parallel chords BD′, BE′, BF′, drawn from the upper extremity of the same diameter. The same thing would also appear immediately from equation $t = l\sqrt{\frac{2}{gh}}$, which shows that the times of descent down different planes are as the lengths divided by the square roots of the heights. But the chords of the arches of the same circle are as the square roots of their verse sines, *i. e.* the arches being measured from either extremity of the vertical diameter, the chords are proportional to the square roots of their heights.

The relation between the space and velocity is given by equation (*f*). Wherefore, if the velocity acquired by descending down the entire length of the plane is sought, this is had by putting *l* for *s* in that equation, which then becomes

$$v = \sqrt{2gh}.$$

This is also the expression for the velocity acquired by falling down h, the vertical height of the plane; from which it appears, that the velocities acquired by falling down different planes, are as the square roots of their heights; and that if the heights are equal, the velocities acquired are also equal.

It also follows, that the velocities acquired by falling through the chords of the several arches of the same circle are proportional to the chords themselves, the circle being placed in a vertical plane, and the chords being drawn to the extremity of the vertical diameter, for the velocities are proportional to the square roots of the heights, and therefore to the chords.

4. There are several other constant forces in nature, which render this theory of practical importance. Of this kind, is the force of resistance arising from tenacity. For when a projectile penetrates a solid substance, the whole depth of penetration, s, being divided into an indefinite number of equal parts ds, the tenacity in each of these parts is the same, and the decrement of velocity produced by its resistance is proportional to the time of its action, *i. e.* to the time taken by the projectile to pass through each of these spaces; and, therefore, the retardation produced by any number of such parts, *i. e.* the retardation suffered by the projectile in passing through any portion of the whole depth of penetration, is simply as the time. This is confirmed by experiment, which shows, that the depths to which a projectile penetrates a bank of earth, or plank of timber, are as the squares of the velocities of projection. This relation between the spaces described, and the velocities destroyed, is the indication of a uniformly retarded velocity; and, therefore, of a constant force.—See equation 3.

This being admitted to be the nature of the force of resistance arising from tenacity, it is easily seen why a bullet

may be sent through a plank without overturning it; or through a door without moving it on its hinges. For agreeably to the notation before adopted, let f be the constant force of resistance; s the space through which it acts, and which, in this case, is the thickness of the plank; v' the velocity of projection; and v that with which the projectile escapes from the remote surface; there is by equation (3)

$$v'^2-v^2=2f.s.$$

a constant quantity: wherefore, $(v'-v)\ (v'+v)$ is constant; shewing, that $v'-v$ varies reciprocally as $v'+v$. But $v'-v$ is the velocity lost, and this is proportional to the motion communicated. Wherefore, the greater the sum of the velocities with which the projectile enters and escapes, the less the motion communicated by it.

5. When the action is that of a variable force, the rate at which the velocity increases or diminishes, being as the acting force, is no longer constant, *i.e.* the velocity is no longer uniformly accelerated, or uniformly retarded: and the change made in that velocity, during a given time, is no longer the measure of the force acting at the beginning of that time. If the force increases, the change of velocity is greater than that due to the uniform action of the force whose expression is sought; and if the force diminishes, the change of velocity is less than that by which the same should be measured. Now, to find the expression for such a varying force at any one instant, it is to be observed, that the change made in the velocity during any time t, is intermediate between the changes that would be effected in the same time, by the forces acting at the beginning and end of that time, *i. e.* putting v', v for the velocities at the beginning and end of the time t, f' and f for the forces; the greater velocity and greater force being denoted by the accented letters: there is

$$f.t<(v'-v)<f'.t$$

i. e. $v'-v$ is intermediate between ft and $f't$. But by di-

minishing the time t indefinitely, the forces f and f' at the beginning and end of that time approach within any assignable difference; and therefore, *a fortiori*, $v'-v$, intermediate between them, shall differ from ft by a quantity less than any assignable: therefore, denoting the indefinitely small increments of the time and velocity by dt and dv, there is $dv=fdt$ or

$$f=\frac{dv}{dt}. \qquad (8)$$

This theorem is demonstrated similarly for a decreasing force.

By Sect. 1. Art. 3. $v=\frac{ds}{dt}$. and therefore, $\frac{d^2s}{dt}=dv=f.dt$. whence

$$f=\frac{d^2s}{dt^2}. \qquad (9)$$

in which the force is expressed as a function of the space and time.

Also, multiplying equation (8) by the equation $\frac{ds}{dt}=v$. we have $f.ds=v.dv$, or,

$$f=\frac{v.dv}{ds}. \qquad (10)$$

whereby the force is represented as a function of the space and velocity.

The theorems contained in equations (8) and (10) may be represented geometrically; the former, by taking a base line AB, with its subdivisions, to represent the time and its subdivisions, (Fig. 136.) and erecting perpendiculars at indefinitely small intervals, to represent the forces acting at the several moments of time, corresponding to the points of the base at which they are raised. Each areola shall then represent the increment of the velocity, or dv. gained in the time represented by its base; and, therefore, the area,

standing on any portion of the base, shall represent the velocity generated in the corresponding portion of time.

The force is never given as a function of the time, and, therefore, the preceding construction is of no use but for illustration.

The force is most commonly known as a function of the space, and, therefore, equation (10) admits of a more useful construction.

Let the same line AB represent the space described by the moving body; and the equidistant ordinates, the forces acting at the points at which they are raised. Then, each areola representing $f.ds$, shall represent its equal, viz. $v.dv$. which is half the increment on the square of the velocity. Wherefore, the change on the square of the velocity, during the motion through any portion of the space AB, shall be represented by twice the area standing on that portion. And thus, the determination of the velocity at any point of the space, is reduced to the quadrature of curves.

The velocity being thus found as a function of the space, may be substituted for v, in equation $v = \frac{ds}{dt}$, whence, by integration, the time of describing any portion of the space is found.

In general, it is to be observed, that each of the equations (8), (9), (10) involves three quantities: and that when the force is given as a function of the space or velocity, one of these may be eliminated, and an equation thus obtained between the remaining two. This proceeding shall now be illustrated by a few examples.

6. The force being supposed to be directed to a fixed centre, and to vary as some power, n, of the distance from that centre, let it be proposed to find the law of the motion of a body ascending or descending in a right line passing through that point.

In general, if x denote the variable distance of the body

from the centre of force, then $ds = -dx$; the body being supposed to descend towards that point. Equation (10), therefore, becomes $vdv = -fdx$, or integrating

$$v^2 = -2\int fdx + \text{C}.$$

Now in the case proposed, if μ denotes the intensity of the force at the unit of distance, the value of the force at any distance x, is μx^n. Substituting this value of f in the equation just obtained, it becomes

$$v^2 = -\frac{2\mu x^{n+1}}{n+1} + \text{C}.$$

But v' denoting the velocity at the distance x', we have

$$v'^2 = -\frac{2\mu x'^{n+1}}{n+1} + \text{C}.$$

And subtracting,

$$v^2 - v'^2 = \frac{2\mu}{n+1}\left(x'^{n+1} - x^{n+1}\right).$$

This equation will equally serve in the case of a body projected in a direction opposite to that of the force. For in that case, the signs of dx and dv are together changed; whereby the differential equation remains unchanged.

The total height of ascent when the body is projected in the direction opposite to that of the force, is had by making $v = o$ in the last equation. The force at the distance x' is $\mu x'^n$, and if h denote the height through which the body must fall under the influence of that force supposed constant, in order to acquire the velocity v', we have

$$x^{n+1} - x'^{n+1} = (n+1)x'^n h.$$

whence

$$x = x'\left[1 + (n+1)\frac{h}{x'}\right]^{\frac{1}{1+n}}.$$

The velocity being expressed as a function of the distance, the time is to be obtained from the formula

$$t=-\int\frac{dx}{v}.$$

To apply these results to different suppositions respecting the law of the force.

For $n=1$. The equation between the space and velocity is

$$v^2-v'^2=\mu(x'^2-x^2).$$

If the body descends from a state of rest, there is

$$v=\sqrt{\mu(x'^2-x^2)}.$$

The velocity with which the body arrives at the centre of force is had by making $x=o$. in the last equation, which gives

$$v=x'\sqrt{\mu}.$$

The body having arrived at this point shall proceed in the same direction to the distance x' with a velocity retarded in the same manner as it was before accelerated: the velocities at equal distances on both sides being equal. Having attained to this distance beyond the centre of force, the body shall return, describing the space $2x'$ with a velocity varying according to the same law; and thus is shall for ever go and return through this space.

The time of descent to any distance x measured from the centre of force is obtained by putting for v, its value in the equation $v=-\frac{dx}{dt}$. which gives

$$dt=-\frac{dx}{\sqrt{\mu(x'^2-x^2)}}.$$

and by integration

$$t=\frac{1}{\sqrt{\mu}}.\text{ arc }\left(\cos.=\frac{x}{x'}\right)$$

which needs no correction if the time be computed from the instant when the body is at the extremity of its space of vibration, *i. e.* when $x=x'$.

The whole time of descent is had by making $x=o$. which gives cos.$=o$. and the arc a quadrant. Wherefore, putting T for this time its value is given by the equation

$$\text{T}=\frac{\pi}{2\sqrt{\mu}}.$$

This being independent of the distance from which the body begins to descend, it follows that bodies arrive at the centre of force in equal times from whatever distances they may have fallen.

By reversing the general formula, we get

$$x=x'\cos.(\sqrt{\mu}.t).$$

for the distance corresponding to any time t; and by substituting this value of x in the equation between the space and velocity, we obtain an equation between the velocity and time.

To represent these results geometrically, let a circle be described (Fig. 137.) whose centre O is the centre of force, and radius OA equal to x'; and at the distance On, equal to x, let the ordinate nm be raised. It is obvious that $nm = \sqrt{x'^2-x^2}$. so that

$$v=\sqrt{\mu}\ nm.$$

The velocity therefore varies as the ordinate. Also, the arc

$$\text{A}m=x'\,\text{arc}\left(\cos.=\frac{x}{x'}\right);$$

and consequently

$$t=\frac{\text{A}m}{\sqrt{\mu}\,x'}.$$

The denominator of this expression is the measure of the velocity acquired in falling to the centre. Therefore, the time is equal to that of describing the arc Am, with that velocity continued uniform.

Forces varying according to this law frequently offer

themselves for consideration. Thus, if the density of the earth were uniform throughout its mass, the force of gravity beneath the surface would follow the law of the simple distance from the centre, *i. e.* at the distance of half the radius, the weight of the body would be reduced to half; at the distance of a third part of the radius, to one-third, and so forth. In this case, the gravity at the surface being denoted by g, there is $g = \mu r$, and substituting in the expressions already obtained, the velocity acquired in descending to the centre is

$$v = \sqrt{gr}.$$

and the time is

$$\mathrm{T} = \frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

The quantities so expressed are readily computed; for if in round numbers, r is 4000 miles, or 21,120000 feet, g being 32.

$$gr = \sqrt{675,840000.} \text{ feet} = 26000 \text{ feet. } q.p.$$

i. e. about five miles. Accordingly the body would pass through the centre, moving at the rate of about five miles per 1″.

For the time of descent to the centre, we have $\pi = \frac{22}{7}.q.p.$ and therefore,

$$\mathrm{T} = \frac{\pi}{2}\sqrt{\frac{r}{g}} = \frac{11}{7}\sqrt{\frac{21,120000''}{32}} = 21\tfrac{1}{3}' \text{ nearly.}$$

The force of elastic bodies is known to follow the law of the distance from the point of rest. Thus the middle point of a musical cord is accelerated or retarded by a force varying as the distance from the middle point of the space described by its motion, or from the point of rest. The theorems here established may, therefore, be applied to the solution of questions relative to the motion of such bodies.

For example, let a bow be supposed to require a weight of 100 pounds applied to its string, to draw it through a space of 30 inches, and that the arrow weighs one ounce; the velocity with which it is discharged is had by putting their particular values for x' and μ in the equation $v = x'\sqrt{\mu}$. Now the force at x', *i. e.* $\mu x'$ is 1600 times the force of gravity, or 1600×32. and $x' = 2\frac{1}{2}$ feet. Therefore $v = 350$ feet per 1″. nearly; or about the fifth part of the velocity with which a ball is discharged from a well loaded musket.

If the force varies inversely as the distance from the fixed point, or $n = -1$. the solution given above fails; and we must have recourse to the general equation

$$v^2 = -2\int f dx + c.$$

From which, by substituting for f its value $\frac{\mu}{x}$, we find

$$v^2 = -2\mu.\log.x + c.$$

And

$$v'^2 = -2\mu.\log.x' + c.$$

Wherefore, the law of the movement is given by the equation

$$v^2 - v'^2 = 2\mu.\log.\frac{x'}{x}.$$

The body being projected in a direction opposite to that of the force, the total height of ascent is had by making $v = 0$. which gives

$$v'^2 = 2\mu.\log.\frac{x}{x'}.$$

whereby it appears that the body under the influence of a force acting according to this law, cannot be sent to an infinite distance, by any finite velocity of projection.

If the body descends from a state of rest, the expression for the velocity is given by the equation $v^2 = 2\mu.\log.\frac{x'}{x}$. At

the centre of force, $x = 0$. and, therefore, the velocity at this point is infinite.

If the force varies inversely as the square of the distance, then, $n = -2$, and the equation expressing the relation between the velocity and space is

$$v^2 - v'^2 = 2\mu\left(\frac{1}{x} - \frac{1}{x'}\right).$$

If the body descends from a state of rest, $v' = 0$. and $v^2 = 2\mu\left(\frac{1}{x} - \frac{1}{x'}\right)$. From which it appears, that in the descent, the velocity varies as the square root of the space described, divided by the square root of the distance from the centre.

For the centre, $x = 0$. wherefore, the velocity at the centre is infinite.

By substituting for v, its value in the equation $v = -\frac{dx}{dt}$, there is

$$dt = -\frac{x^{\frac{1}{2}}.dx}{\sqrt{\frac{2\mu}{x'}}\,(x'-x)^{\frac{1}{2}}} = -\frac{xdx}{\sqrt{\frac{2\mu}{x'}}\,(xx'-x^2)^{\frac{1}{2}}}.$$

and integrating,

$$t = \sqrt{\frac{x'}{2\mu}}\left\{(xx'-x^2)^{\frac{1}{2}} + \tfrac{1}{2}x' \operatorname{arc}\left(\cos. = \frac{2x-x'}{x'}\right)\right\}.$$

which requires no correction.

To represent these things geometrically, let A be the point from which the body begins its descent towards O, the centre of force, (Fig. 138.) and let the line AO be bisected at C: then $x' =$ AO, and the body being at m, $x - \frac{1}{2}x' =$ Cm. Moreover, if a circle is described with the centre C and radius CA, and an ordinate mn, is raised at m, we shall have

$$\frac{x'}{2} \text{ arc } \left(\cos. = \frac{x - \frac{1}{2}x'}{\frac{1}{2}x'}\right) = \text{arc. A}n. \qquad \text{and } (xx' - x^2)^{\frac{1}{2}} = mn.$$

Now, $\frac{1}{4}$ AO $\times$ arc An = sector CAn; and $\frac{1}{4}$ AO $\times$ mn = triangle CnO. Wherefore, the time of descent to any point as m, is proportional to the mixed-lined area OAn, contained by the right lines OA, On, and the circular arc An.

This construction is given by Newton.

When the body arrives at the centre, there will be $x = o$. and

$$\mathrm{T} = \frac{\pi}{2} \frac{x' \sqrt{x'}}{\sqrt{2\mu}}.$$

which shows that the times of descent from different distances to the centre of force, vary in the sesquiplicate ratio of those distances.

10. To determine the law of the vertical movement of a body in a fluid medium. The fluid being supposed to be without tenacity, the resistance opposed to the motion of the body shall be measured by the motion communicated to the fluid, *i. e.* by the quantity displaced in a given time, multiplied by the velocity imparted to it. Wherefore, the resistance is as the square of the velocity, and may be denoted by mv^2. the coefficient m being the resistance to the body moving with a unit of velocity. Its value depends on the form of the body and the density of the medium, and is, therefore, constant, when both of these are given. Moreover, in problems of this nature, the spaces described being small, in relation to the distance from the centre of the earth, the force of gravity shall be supposed to be constant.

If the body is projected vertically upwards, its motion is retarded, both by the force of gravity and by the resistance of the medium; and the whole force then acting on the body is $-(g + mv^2)$. But if the body descends, gravity acts as an accelerating force; and this being diminished by the resis-

tance of the medium, the force with which it is effectively accelerated, is $g-mv^2$.

To begin with the case of an ascending body, we have $f=-(g+mv^2)$. and this value of the force being substituted in the general equations, $fdx=v.d.v$, $fdt=dv$, we have

$$dx=-\frac{v.dv}{g+mv^2}. \qquad dt=-\frac{dv}{g+mv^2}.$$

These equations being integrated, give

$$2mx=\mathrm{c}-\log.(g+mv^2).$$

$$\sqrt{mg}.t=\mathrm{c}'-\mathrm{arc}\left(\tan.=\sqrt{\frac{m}{g}}.v\right).$$

The constants are determined by observing that for $x=0$, or for $t=0$, the value of v is the velocity of projection. Wherefore, putting v', as before, for that velocity,

$$\mathrm{c}=\log.(g+mv'^2). \qquad \mathrm{c}'=\mathrm{arc}\left(\tan.=\sqrt{\frac{m}{g}}.v'\right).$$

Wherefore,

$$2\,mx=\log.\left(\frac{g+mv'^2}{g+mv^2}\right).$$

$$\sqrt{mg}.t=\mathrm{arc}\left(\tan.=\sqrt{\frac{m}{g}}.v'\right)-\mathrm{arc}\left(\tan.=\sqrt{\frac{m}{g}}.v\right).$$

These two equations contain the complete solution of the problem for an ascending body, since by exterminating v, they give an equation between the space and time.

For the total ascent, $v=0$. Accordingly putting x' for this height, and t' for the time of total ascent, we shall have

$$x'=\frac{1}{2m}.\log.\left(1+\frac{mv'^2}{g}\right).$$

$$t'=\frac{1}{\sqrt{mg}}.\mathrm{arc}\left(\tan.=\sqrt{\frac{m}{g}}.v'\right).$$

When the body arrives at the highest point of its flight, it shall return as from a state of rest; and the force with which it is accelerated in its descent being $g - mv^2$, the equations by which this part of the problem is to be solved, are

$$dx = \frac{v.dv}{g-mv^2}. \qquad dt = \frac{dv}{g-mv^2}.$$

The former of these, by integration, gives

$$2mx = \text{c} - \log.\,(g - mv^2). \qquad (11)$$

And since x is now measured from the point where $v = 0$, $\text{c} = \log.\,g$. Wherefore,

$$2mx = \log.\left(\frac{g}{g-mv^2}\right).$$

To prepare the second of the differential equations for integration, it may be put in this form,

$$dt = \frac{1}{2\sqrt{g}}\left\{\frac{dv}{\sqrt{g}+v.\sqrt{m}} + \frac{dv}{\sqrt{g}-v\sqrt{m}}\right\}.$$

which, by integration, gives

$$2\sqrt{mg}.t = \log.\left\{\frac{\sqrt{g}+v.\sqrt{m}}{\sqrt{g}-v.\sqrt{m}}\right\}. \qquad (12)$$

This requires no correction, since $t = 0$, gives $v = 0$.

For x infinite in equation (11), or t infinite in equation (12), there is

$$v = \sqrt{\frac{g}{m}}.$$

This, therefore, is the limit to which the velocity of the descending body always approaches, without ever attaining to that value.

When the body thrown upwards with the velocity v', returns to the point from which it was projected, it is evident that it does not regain the whole of this velocity, inas-

much as in each point of the space, the force which accelerates the motion in the descent, is less than that by which the motion upwards was retarded. These velocities are compared by equating the values of x' and x, as given for the body in its ascent and descent. This gives

$$\frac{g+mv'^2}{g}=\frac{g}{g-mv^2}. \quad \text{and } v^2=\frac{g.v'^2}{g+mv'^2}.$$

Wherein mv'^2 is the resistance encountered by the body moving with the velocity of projection. The same statement will serve for comparing the velocities of ascent and descent at any point, by putting in place of v', the velocity of ascent at that point.

SECTION III.

OF FORCES PRODUCING CURVILINEAR MOTION.

1. When a body moves in a curve of any kind, the direction of its motion is continually changed, which cannot be, unless the body is under the influence of a force or forces acting in directions different from that of the motion. Now it is the business of mechanical philosophy to connect these forces and movements, so that each may be ascertained, as to its intensity, direction, and laws, when the other is given. By those who first cultivated this branch of mechanics, such questions were managed by resorting to the indefinitely small movements successively compounded, and the forces by which they were produced. Thus, if ac (Fig. 139.) is the path of the body; am the portion of the tangent at a, which would have been described in a unit of time with the velocity at that point; and ao, the line actually described in that time: then, an, equal and parallel to mo, is the space which would have been described, had the force been applied to the body at rest. Moreover, the unit of time, and therefore the lines ao, an, being taken indefinitely small, the force during that time is regarded as unchanged in quantity and direction: and, consequently, $2an$, or $2mo$, is the measure of the velocity due to the action of the force; and, therefore, also the measure of the force itself. In the same way, $a'm'$ being the portion of the tangent to the curve at a' which would be described in a unit of time with the velocity at that point; and $a'o'$ the space actually described in that time; twice $m'o'$, or twice $a'n'$, to which it is equal and parallel, is the measure of the force at a'. Accordingly,

the deflecting force at a, is to that acting at a', as mo to $m'o'$, the times of describing the elementary arcs ao, $a'o'$, being equal; or if the times are unequal in any ratio, the forces are as the same lines divided by the squares of the times, agreeably to what has been established relative to right lined motions produced by the action of constant forces. Now, when the curve is given, as also the law of the motion in that curve, it is the business of geometry to ascertain the relation of the lines mo, $m'o'$, and, therefore, of the forces measured by those lines; likewise, when the velocity and direction of the movement at any one point of space are given, together with the intensity and law of the force by which the body is subsequently influenced, it belongs to the same science to investigate the nature of the curve or trajectory. The former of these was called the direct, and the latter the inverse problem in this branch of mechanics.

2. This method, which was somewhat circuitous, has given place to others, according to which, the forces and movements are reduced to rectangular or polar coordinates. Of these, the former, as being more simple as well as more extensibly applicable, is that which shall be treated of in the present section. In this method, as already intimated, the forces, whatever their number or directions, are reduced to three forces acting in directions parallel to three rectangular axes, after the manner explained in 'Statics,' Sect. I. Art. 9, &c. Likewise, the material point being projected on the same axes, those projections are regarded as so many points which accompany the material point in its motion through space. In this way, the movement of the material point is resolved into three movements, which are those of its projections: and as the movement, parallel to any one axis, is not influenced by the forces parallel to the other axes, it follows, that each of the decomposed motions may be regarded as being influenced solely by the force in its

direction; whereby the theory of forces producing curvilinear motion, is reduced to that of three forces producing so many right lined motions.

3. To illustrate this method by example, let a body be supposed to be projected in a line making any angle with the vertical, whilst it descends in the vertical with a velocity uniformly accelerated.

Let BD (Fig. 140.) be the space that would be described in a given time with the velocity of projection; and BA, taken in the vertical, that described in the same time, with the velocity uniformly increasing. Then at the end of the same time, the body shall be brought by the compound motion to C, the extremity of the diagonal of the parallelogram, whose sides are BD, BA. Dividing BD into a number of equal parts, Bo, oo', $o'o''$, o''D, the times of reaching the points o, o', o'', D, with the velocity of projection, are as the numbers 1, 2, 3, 4, whilst the spaces on, $o'n'$, $o''n''$, DC, described in the vertical, are as squares of those numbers; *i. e.* as 1, 4, 9, 16. And such, generally, is the relation between the lines on, $o'n'$, $o''n''$, &c. and the lines Bo, Bo', Bo'', &c. or between Bm, Bm', Bm'', &c. and mn, $m'n'$, $m''n''$, &c. the former being as the squares of the latter; wherefore, the curve described is the parabola. Now, without adverting to the properties of this curve, or embarrassing ourselves with the consideration of the compound motion, all the problems respecting the time of flight, the range, the height of ascent, &c. may be solved, by considering the motions as subsisting separately, and determining the question for each of these movements.

It is evident that the motion shall be in the plane of the rectilinear movements, seeing that there is no reason why the projectile should deviate from this plane, to one side rather than the other. Wherefore, it will be sufficient to refer to two axes in that plane, and for the sake of simplicity, they may be drawn through B, the point of departure: one of them vertical and the other horizontal.

The uniformly accelerated velocity, being parallel to the axis of y, is not to be resolved: this velocity being proportional to the time, may be denoted by gt. The velocity of projection, when oblique to the horizon, is to be resolved into two, of which one is vertical, and the other horizontal. Let v' denote this velocity of projection; and e the elevation of the line of direction, or the angle which it makes with the horizon. Then, $v'.\cos.e$, $v'.\sin.e$, are the velocities of projection estimated in directions parallel to the axes of x and y respectively. The motion parallel to the axis of x, is, therefore, $v'.\cos.e$: and the vertical movement of the projectile being the difference between the uniform motion of projection upwards, and the uniformly accelerated motion downwards, is $v.\sin.e-gt$.

The altitude to which the projectile ascends, and the time in which it returns to the axis of x, depend solely on the vertical movement. These are given by equations (4) and (6), by making $v=0$, using g with a negative sign, and changing v' for $v'.\sin.e$. In this way, the latter equation gives for the height of ascent,

$$s=\frac{v'^2.\sin.^2e}{2g}.$$

and the former, for the time,

$$t=\frac{v'.\sin.e}{g}.$$

Twice this time is that of the ascent and subsequent descent, or as it is commonly called, the time of flight.

To find the horizontal range, the horizontal velocity is to be multiplied by the time of flight. Wherefore, putting R for the horizontal range, its value is given by the equation

$$\mathrm{R}=v'.\cos.e\times\frac{2v'.\sin.e}{g}=\frac{v'^2.\sin.2e}{g}.$$

Hence it appears, that when the velocity of projection is

given, the amplitude or range varies as the sine of twice the elevation: and that it is, therefore, a maximum, when the elevation is 45°. Also, that for two elevations which are complements, one of the other, the horizontal ranges are equal.

The place of the projectile for any time t, is found by ascertaining its coordinates for that time. The horizontal velocity is uniform; therefore, $x = t.v' \cos. e$. But the ordinate y, is the difference of the spaces described by two vertical motions. One of these motions, viz. $v'.\sin.e$, is uniform; and, therefore, the space described by this motion is $t.v'.\sin.e$. The other is the uniformly accelerated motion, whose value for any time t, is gt, and the space described by this motion is $\frac{g.t^2}{2}$. Accordingly,

$$y = t.v'.\sin.e - \frac{g.t^2}{2}.$$

If the problem is to find the velocity of projection, or the elevation requisite for striking a given scope, it is evident that the value of t must be the same in the equations $x = t.v'.\cos.e.$ and $y = t.v'.\sin.e - \frac{g.t^2}{2}$. Wherefore, the values of t, as given by those equations, are to be put equal, *i. e.* t is to be exterminated between them, whence

$$y = x.\tan.e - \frac{g.x^2}{2v'^2.\cos.^2 e}.$$

This equation becomes somewhat more simple, by putting for v'^2, its value, given by the equation $v'^2 = 2gh$; in which, h denotes the height from which a body should fall, in order to acquire the velocity v', or, as it is called, the height due to that velocity. Making this substitution, the equation is

$$y = x.\tan.e - \frac{x^2}{4h.\cos.^2 e}.$$

This is the equation of the curve described by the projectile: from which it appears again, that the curve is a parabola, whose axis is parallel to the axis of y. Also, that $4h.\cos.^2 e$, is its principal parameter; and $4h$ the parameter of the diameter passing through the point of projection. By this equation, any one of the four quantities, x, y, e, h, is ascertained when the remaining three are given. Thus, if the scope is given by its coordinates x', y'; and if the angle of elevation is also given, or $e=e'$; these values being substituted in the equation of the curve, we have

$$h=\frac{x'^2}{4.\cos.^2 e'(x'.\tan.e'-y')}.$$

or, because $h=\dfrac{v^2}{2g}$.

$$v=\frac{x'\sqrt{\frac{1}{2}g}}{\cos.e'\sqrt{(x'.\tan.e'-y')}}.$$

If $y'=x'.\tan.e'$, the value of v is infinite; and when y' is greater, the value of v is impossible: which is, indeed, evident, inasmuch as the curve must fall below its tangent.

If it is required to find the elevation, in order to reach a given scope with a given velocity of projection, it will be convenient to express $\cos.e$ by means of $\tan.e$.

Substituting in the equation of the curve for $\dfrac{1}{\cos.^2 e}$, its value, which is $1+\tan.^2 e$. and solving for $\tan.e$, there is

$$\tan.e=\frac{2h'\pm\sqrt{4h'^2-4h'y'-x'^2}}{x'}.$$

From which it appears, that the scope cannot be reached with the given velocity of projection, if $4h'y'+x'^2$ exceeds $4h'^2$, *i. e.* if h' is less than $\dfrac{y'+\sqrt{y'^2+x'^2}}{2}$. But that when h' exceeds this value, there are two values of e, between whose tangents the quantity $\dfrac{2h'}{x'}$ is an arithmetical mean.

4. The case here adduced, when treated as above, affords an exemplification of the use of the principles stated in the preceding Art.; and as in this case, the law of the motion in each of the axes was given, there could be no difficulty in finding for any given time, the place of the body, its velocity, or the direction of its motion. But in most cases, the direction of the force, with respect to the axes, is continually changed; and in such cases, it is evident, that the reduction of the forces and motions to the same axes must first be assumed, in order that by the connexion between them, we may proceed to the discovery of either from the other.

The relation between the force and velocity in each axis is expressed as follows:

The velocities in the directions of the axes are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. Wherefore, putting, as usual, X, Y, Z for the sum of the forces in the axes of x, y, and z respectively, we have by equation (9) of preceding section,

$$\frac{d^2x}{dt^2} = \text{X}. \qquad \frac{d^2y}{dt^2} = \text{Y}. \qquad \frac{d^2z}{dt^2} = \text{Z}. \qquad (1)$$

When the motion is given, those into which it is resolved, and, therefore, the first members of these equations, are given; whence the resolved forces are known, and, therefore, their resultant, which is the square root of the sum of the squares of its rectangular components. And conversely, when the force is given, those into which it is resolved, *i. e.* the second members of those equations are given; and, therefore, the first members, which by integration give the resolved movements; and these, being compounded, give the velocity in absolute space.

To find the path of the motion, equations (1) are to undergo a second integration, and at each integration there are introduced three constant quantities: those introduced

by the first integration, are the resolved velocities corresponding to certain coordinates, and expressed as at the beginning of this article; and the constants, introduced by the second integration, are those coordinates themselves. The times being eliminated from the three integrals, and the forces being functions of the coordinates x, y, and z, the two resulting equations express the relations among those coordinates, and are therefore the equations of the curve. The three differential equations are requisite, when the path of the movement is a curve of double curvature; but when the motion is limited to a plane, that plane may be taken for the plane of the axes of x and y, and then $z = 0$, $\text{Z} = 0$, and the differential equations relative to the other two axes being integrated, and t eliminated, there results a single equation between the coordinates x and y, which is that of the plane figure.

3. To proceed according to this method, let the equations (1) be multiplied, the first by dx, the second by dy, and the third by dz, we have

$$\frac{dx.d^2x}{dt^2} = \text{X}.dx. \qquad \frac{dy.d^2y}{dt^2} = \text{Y}.dy. \qquad \frac{dz.d^2z}{dt^2} = \text{Z}.dz.$$

Adding these equations, there is

$$\frac{dx.d^2x + dy.d^2y + dz.d^2z}{dt^2} = \text{X}.dx + \text{Y}.dy + \text{Z}.dz. \qquad (2)$$

The first member of this equation is $\frac{d(dx^2 + dy^2 + dz^2)}{2dt^2} =$ $\frac{d(ds^2)}{2dt^2} = \frac{ds.d^2s}{dt^2}$. Wherefore,

$$\frac{ds.d^2s}{dt^2} = \text{X}dx + \text{Y}dy + \text{Z}dz. \qquad (3)$$

From equation (3) some important conclusions are derived. For putting it in this form,

$$\frac{d^2s}{dt^2} = \text{X}.\frac{dx}{ds} + \text{Y}.\frac{dy}{ds} + \text{Z}.\frac{dz}{ds}. \qquad (4)$$

As the coefficients $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, are the cosines of the angles at which ds, the element of the curve, is inclined to the axes of x, y, and z, respectively, it follows that the second member of this equation expresses the sum of the forces, reduced each to the direction of the tangent. Accordingly, putting S for the tangential force, the equation is $\frac{d^2s}{dt^2} = \mathrm{X}.\frac{dx}{ds} + \mathrm{Y}.\frac{dy}{ds} + \mathrm{Z}.\frac{dz}{ds} = \mathrm{S}$. Whence it appears, that the tangential force is expressed by the second differential coefficient of the arc with respect to the time; an expression strictly analogous to that of a force producing right lined motion.

Equation (3) put in this form $\frac{d(ds^2)}{dt^2} = 2.(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz.)$ gives by integration,

$$v^2 = 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) + \mathrm{C}. \qquad (5)$$

The value of the quantity within the sign of integration is $\mathrm{S}.ds.$ and this value being substituted, the equation is

$$v^2 = 2\int \mathrm{S}.ds + \mathrm{C}.$$

From which it follows, that if the resultant of the forces is resolved into two, viz. in the direction of the tangent, and in that of the normal to the curve, the velocity shall be affected solely by the tangential force: and as this force can have no effect in changing the direction of the motion, it likewise follows, that the change of direction is effected, exclusively by the normal force.

The last equation is particularly useful when the tangential force is given as a function of the arc, of which a remarkable example will occur when constrained motion is treated of.

When the quantity $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ is an exact differ-

ential of a function, $\phi(x, y, z)$, of the three coordinates, equation (5) becomes

$$v^2 = 2\phi(x, y, z) + \text{C};$$

And if v' denote the velocity at any point of the trajectory whose coordinates are x', y', z',

$$v'^2 = 2\phi(x', y', z') + \text{C}.$$

Wherefore,

$$v^2 - v'^2 = 2\phi(x, y, z) - 2\phi(x', y', z').$$

In this case, then, the difference between the squares of the velocities in any two points of the path, is independent both of the form of the curve between those points, and of the time in which it is described; or in other words, the velocity being given for any point of the path, it is also given for any other point; and this, independently of the figure contained between those points, and of the time taken to describe it.

This will be the case when the acting force is directed to a fixed centre, and is at the same time a function of the distance from that centre; or when it is the resultant of two or more other forces of this nature: and then the expression $\text{X}.ds + \text{Y}.dy + \text{Z}.dz.$ is reducible to another, wherein the variables are separated.

If p denote the distance of the moving point from one of these fixed centres, and α, β, γ, the angles contained by that line with the three axes, $\frac{dp}{ds}$ will be the cosine of the the angle contained by the same line with the element of the curve; and $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, being the cosines of the angles made by the element of the curve with the three axes, there is

$$\frac{dp}{ds} = \cos.\alpha\frac{dx}{ds} + \cos.\beta\frac{dy}{ds} + \cos.\gamma\frac{dz}{ds}.$$

Let both members of this equation be multiplied by $\text{P}ds$, P

denoting the intensity of the force directed to the fixed centre, and it becomes

$$\text{P}dp = \text{P}(\cos.\alpha dx + \cos.\beta dy + \cos.\gamma dz).$$

There will be a similar equation for each of the other forces, so that we have by addition

$$\Sigma(\text{P}dp) = dx\,\Sigma(\text{P}\cos.\alpha) + dy\,\Sigma(\text{P}\cos.\beta) + dz\,\Sigma(\text{P}\cos.\gamma).$$

the symbol Σ denoting the sum of all the products similar to those to which it is prefixed. But

$$\Sigma(\text{P}\cos.\alpha) = \text{X}. \qquad \Sigma(\text{P}\cos.\beta) = \text{Y}. \qquad \Sigma(\text{P}\cos.\gamma) = \text{Z}.$$

so that

$$\text{X}dx + \text{Y}dy + \text{Z}dz = \Sigma(\text{P}dp).$$

It appears then that when there are several forces directed to fixed centres, which are moreover functions of the distances from those centres, the velocity being given for any one point of the path, is given for any other point; and this independently of the form of the curve described between those points, and of the time taken to describe it.

If there is but one such force acting on the body, the proposition is true, not only for the transit between two given points, but also between two given distances from the centre of force. For in this case,

$$v^2 = 2\int \text{P}.dp + \text{C} = 2\phi(p) + \text{C}.$$

From this equation it appears, that if two bodies, influenced by the same accelerating force directed to a fixed point, have equal velocities at any equal distances from that point, their velocities, at any other equal distances, shall be equal, *i. e.* several spherical surfaces being described round the centre of force, if the bodies have equal velocities at any one of those surfaces, they shall arrive at any other of the surfaces with equal velocities.

The geometrical proof of this proposition is exceedingly simple. For let ds, ds' be the elements of the two curves contained between the spheric surfaces, whose radii are r,

and $r+dr$. The tangential forces are $\text{P}\frac{dp}{ds}$. and $\text{P}\frac{dp}{ds'}$. *i.e.* inversely as the elements ds, ds'. But the changes made in the squares of the velocities by describing the spaces ds, ds' are expressed by the double products of those forces and the arcs, or by $2\text{P}dp$, and are the same for both curves.

This is the 40th Prop. of the first Book of the *Principia Math.* of Newton.

4. Having the equations $\frac{d^2x}{dt}=\text{X}.dt.$ $\frac{d^2y}{dt}=\text{Y}.dt.$ $\frac{d^2z}{dt}=\text{Z}.dt.$ if the first is multiplied by y, and the second by x, we have by subtraction,

$$\frac{x.d^2y-y.d^2x}{dt}=(x\text{Y}-y\text{X})\,dt.$$

In the same manner we obtain two other equations

$$\frac{z.d^2x-x.d^2z}{dt}=(z\text{X}-x\text{Z})dt.$$

$$\frac{y.d^2z-z.d^2y}{dt}=(y\text{Z}-z\text{Y})dt.$$

The first members of these equations are the differentials of the quantities

$$\frac{x.dy-y.dx}{dt}, \qquad \frac{z.dx-x.dz}{dt}, \qquad \frac{y.dz-z.dy}{dt}.$$

Wherefore,

$$\frac{x.dy-y.dx}{dt}=c+\int(x\text{Y}-y\text{X})dt.$$

$$\frac{z.dx-x.dz}{dt}=c'+\int(z\text{X}-x\text{Z})dt.$$

$$\frac{y.dz-z.dy}{dt}=c''+\int(y\text{Z}-z\text{Y})dt.$$

$\frac{dy}{dt}$ and $\frac{dx}{dt}$ being the velocities of the moving point, estimated in the directions of the axes y and x, $x\frac{dy}{dt}-y\frac{dx}{dt}$ is the *mo-*

ment, with respect to the axis of z, of the quantity of motion acquired at the end of the time t. The first members of the other two equations, in like manner, are the similar moments with respect to the axes of y and x. On the other hand, since $\mathrm{X}dt$, $\mathrm{Y}dt$, $\mathrm{Z}dt$, are the quantities of motion impressed by the acting forces during the time dt, it is obvious that the quantities under the sign of integration are the moments of these elementary motions taken with respect to the same axes; and the preceding equations show in what manner the moments of the velocity acquired at the end of any time, are made up of the moments of the velocities engendered during each instant by the action of the moving forces.

But the first members of the preceding equations bear another signification. We may suppose the elementary arc, ds projected on the planes of the axes: and the reasoning concerning all these projections being the same, it will be sufficient to consider the projection on one of those planes, which may be that of xy. Accordingly, let o be the origin, (Fig. 141.) $o\mathrm{A}$ the axis of x, and $o\mathrm{B}$ the axis of y, and mn the projection of the elementary arc on the plane of these axes. Then, ma, me, nb, nf, being perpendiculars on those axes, and the former being produced to meet the latter pair at c and h, we shall have $oa = x$, $oe = y$, $mc = dx$, $mh = dy$. Wherefore, $xdy - ydx$, is the difference of the rectangular parallelograms fm, mb. But the point o, lying within the angle ema, contained by the sides of the parallelogram hc, the difference of the rectangles mf, mb is equal to the product of mn, the diagonal of the parallelogram hc, into its perpendicular distance from o, 'Statics,' Sect. 4. Art. 2. or to twice the area of the triangle mon. But this triangle is the projection of that whose base is ds, and whose vertical angle is at o; wherefore, $xdy - ydx$ expresses twice the projection of that area on the plane of xy. The same reasonings apply to the other equations; and we learn that the

numerators in the first members of the preceding equations are double of the projections, on the three coordinate planes, of the elementary area described by the moving point in the time dt.

Or thus: putting r for the distance of the origin o, from the projection of the material point on the plane of x, y; and ω for the angle made by the line r, with the axis of x, we have $x = r.\cos.\omega.$ $y = r.\sin.\omega.$ whence,

$$dx = -r.\sin.\omega.d\omega. \qquad dy = r.\cos.\omega.d\omega.$$

and therefore,

$$x.dy - y.dx = r^2.d\omega.$$

But $r.d\omega$ is the circular arc, described with the centre o and radius r; wherefore, $r^2.d\omega$ is twice the projection of the triangular area whose base is ds, and whose vertex is at the origin o.

When the forces acting on the point are reduced to a single force directed to the origin, the moments of this force, taken with respect to that point, vanish, and we have

$$x\mathrm{Y} - y\mathrm{X} = 0, \qquad z\mathrm{X} - x\mathrm{Z} = 0, \qquad y\mathrm{Z} - z\mathrm{Y} = 0.$$

Consequently, the equations above obtained are reduced to

$$xdy - ydx = cdt,$$
$$zdx - xdz = c'dt,$$
$$ydz - zdy = c''dt.$$

The projections of the elementary areas described during the time dt are therefore constant, and the areas described in any finite time are proportional to that time. Reciprocally, if the areas are proportional to the times, or the latter equations fulfilled, the former must hold good also, and the direction of the force must pass through the origin.

Let the equations last obtained be multiplied, each by the variable which it does not include, and then added; there is

$$cz + c'y + c''x = 0.$$

which is the equation of a plane passing through the origin. This result, therefore, shows that a body urged by a force tending to a fixed point, shall move in a plane passing through that point.

5. It may be satisfactory to the student to see these things proved geometrically; wherefore, let c be the centre of force, (Fig. 142); o the material point, whether moving in the plane of xy, or projected on that plane; and let om be the line it would describe in the time dt, were it not influenced by a force. On this supposition, the triangle ocm would be described by the radius vector in the time dt. But if at o it is urged by a force directed to c, which would carry it through the space on, in the time dt, the line described by the compound motion in the time dt shall be oo', the diagonal of the parallelogram, whose sides are om, on; and the area described by the radius vector shall be the triangle oco'. But mo' being parallel to oc, the triangles oco', ocm, standing on the same base oc, are equal: If the material point were to receive no other impression, it would proceed in the line oo' produced, describing in the time dt the space $o'm'$ equal to oo'; but if at o' it receives an impression, which in that time would carry it through the space $o'n'$, the line described in the time dt shall be $o'o''$, the diagonal of the parallelogram whose sides are $o'm'$, $o'n'$; and the area described by the radius vector shall be o'co''. But the triangles o'co'', o'cm', standing on the same base, and between the same parallels, are equal: and as o'c$m' = o$co', it follows, that o'c$o'' = o$co'.

The triangular areas described round the centre of force in the equal moments dt, being thus proved to be equal, it follows, that the areas described in any portions of time whatsoever, are proportional to the times. Conversely, if the areas described in the time dt are equal, the area o'co'' = area oco' = area o'cm'. Wherefore, the triangles o'co'' and o'cm' standing on the same base, being equal, they must

be within the same parallels; *i. e.* $m'o''$ must be parallel to o'C, and, therefore, the line $o'n'$, in which the force is directed, must pass through the point C. The equable description of areas round any point is therefore the indication of a deflecting force, whose line of direction constantly passes through that point.

Whilst the force is directed to the same point C, the figure of the trajectory must be a plane; for the line $o'o''$, (Fig. 142.) is in the plane of $o'm'$, $o'n'$; *i. e.* in the plane of oo', $o'n'$, or of oCo'; and the same thing is shown, in the same way, of the remaining sides of the polygon. Moreover, in this case, the areas described in equal times, dt, being equal, the velocities are inversely as the perpendiculars from the centre of force on the sides of the polygon. Wherefore, when the polygon terminates in a curve, the velocities at different points are inversely as the perpendiculars from the centre of force on the tangents at those points.

SECTION IV.

OF CENTRAL FORCES.

1. When a material point is urged by a single force tending to a fixed centre, which force is represented by some function of the distance from that centre, the general formulæ, given in the preceding section, are susceptible of simplifications; which, on account of the importance of the case, merit a distinct consideration.

It has been already shewn (Sect. 3. Arts. 4. and 5.) that, in the case under consideration, the point will move in a plane passing through the centre of force, as indeed is apparent; for every thing being similar on both sides of the plane drawn through the direction of the initial velocity and the fixed centre of force, there is no reason why the moveable should deviate from that plane to one side rather than the other; that plane will, therefore, contain the path of the moving point. If then we take this plane for the plane of xy, we may dispense with the third of our fundamental equations, (1.) of last Sect. and the motion of the point will be given by the two equations,

$$\frac{d^2x}{dt^2} = \text{X}. \qquad (1)$$

$$\frac{d^2y}{dt^2} = \text{Y}. \qquad (2)$$

2. As the force by which the moving point is urged varies as a function of the distance of that point from a fixed centre, it is natural and convenient. that its position be referred to that centre by means of polar coordinates. For this purpose, the equations above may be transformed as

follows: if R denote the central force, and ω the angle made by its direction with the axis of x, we shall have the equations

$$\text{X} = -\text{R}.\cos.\omega. \qquad \text{Y} = -\text{R}.\sin.\omega.$$

in which the force R has an opposite sign to that of X and Y, inasmuch as its direction is *towards* the origin. If now we multiply the former of these equations by $\sin.\omega$, and the latter by $\cos.\omega$, and subtract, we have

$$\text{Y}.\cos.\omega - \text{X}.\sin.\omega = 0.$$

Again, multiplying the former by $\cos.\omega$. and the latter by $\sin.\omega$, and adding, we find

$$\text{Y}.\sin.\omega + \text{X}.\cos.\omega + \text{R} = 0.$$

It now remains to substitute for X and Y their values expressed by polar coordinates; these may be obtained as follows: if r denote the distance of the moving point from the origin, or *radius vector*, we have

$$x = r.\cos.\omega, \qquad y = r.\sin.\omega;$$

differentiating these equations twice, all the quantities being considered as variable, we find

$$d^2x = d^2r.\cos.\omega - 2dr.d\omega.\sin.\omega - rd\omega^2.\cos.\omega - rd^2\omega.\sin.\omega.$$
$$d^2y = d^2r.\sin.\omega + 2dr.d\omega.\cos.\omega - rd\omega^2.\sin.\omega + rd^2\omega.\cos.\omega.$$

Therefore, dividing by dt^2, we have, by equations (1) and (2),

$$\text{X} = \left(\frac{d^2r}{dt^2} - r.\frac{d\omega^2}{dt^2}\right)\cos.\omega - \left(2\frac{dr}{dt}.\frac{d\omega}{dt} + r.\frac{d^2\omega}{dt^2}\right)\sin.\omega.$$

$$\text{Y} = \left(\frac{d^2r}{dt^2} - r.\frac{d\omega^2}{dt^2}\right)\sin.\omega + \left(2\frac{dr}{dt}.\frac{d\omega}{dt} + r.\frac{d^2\omega}{dt^2}\right)\cos.\omega.$$

Substituting now these values of X and Y in the equations above, we obtain the equations

$$2\frac{dr}{dt}.\frac{d\omega}{dt} + r.\frac{d^2\omega}{dt^2} = 0. \qquad (3)$$

$$\frac{d^2r}{dt^2} - r.\frac{d\omega^2}{dt^2} + \text{R} = 0. \qquad (4)$$

We have thus transformed equations (1) and (2) into two polar equations, which we shall use hereafter in determining the motion of the point.

3. If we multiply equation (3) by $r.dt$, it becomes

$$2rdr.\frac{d\omega}{dt}+r^2.\frac{d^2\omega}{dt}=d.\left(r^2.\frac{d\omega}{dt}\right)=0.$$

the integral of which is

$$r^2.\frac{d\omega}{dt}=c. \qquad (5)$$

Now, $r^2d\omega$ is twice the differential of the area described round the centre of force; it follows, therefore, from this equation, that the increment of the area described in a given time by the moving point round the centre of force is constant; or, that the areas described are proportional to the times of their description, as has been already shown in the preceding section.

Equation (5) may be thus written,

$$\frac{d\omega}{dt}=\frac{c}{r^2}.$$

from which we conclude that the angular velocity of the moving point varies inversely as the square of the distance from the centre of force; $\frac{d\omega}{dt}$ expressing that angular velocity.

4. The integral of equation (4) may also be obtained with the aid of equation (3). For if equation (4) be multiplied by dr, and equation (3) by $rd\omega$, and the results added together, we shall have

$$dr.\frac{d^2r}{dt^2}+rdr.\frac{d\omega^2}{dt^2}+r^2d\omega.\frac{d^2\omega}{dt^2}+\text{R}dr=0.$$

the integral of which will be

$$\frac{dr^2}{dt^2}+r^2.\frac{d\omega^2}{dt^2}+2\int\text{R}dr=k. \qquad (6)$$

k being the arbitrary constant.

From this equation we immediately obtain the velocity of the moveable at any distance from the centre of force: for, observing that $dr^2 + r^2 d\omega^2 = ds^2$, equation (6) may be written

$$\frac{ds^2}{dt^2} + 2\int \text{R}dr = k.$$

But $\frac{ds^2}{dt^2} = v^2$, v representing the velocity, therefore,

$$v^2 = k - 2\int \text{R}dr. \qquad (7)$$

Comparing this with the expression for the velocity of a point moving in a right line to or from the centre of force, (Sect. II. Art. 6.) we see that they shall be equal, if the constants are equal; it follows, therefore, that if the velocity of the point moving in a curve by the action of a central force, be equal to the velocity of the point moving directly to, or from the centre, at any equal distances, they shall be equal at all equal distances. We need not therefore, in what follows, dwell on the applications of this formula for the velocity to particular laws of force, as this has been sufficiently done in a preceding section, where we treated of the motion of a material point in a right line passing through the centre of force.

As the force R is by hypothesis a function of the distance from the centre, the integral $\int \text{R}dr$ is likewise a function of the same quantity: denoting this by ϕr, equation (7) may be written

$$v^2 = k - 2\phi(r).$$

The constant, k, shall be determined, if the velocity at any given distance be known. For, let v' be the velocity at the given distance r', then,

$$v'^2 = k - 2\phi(r').$$

which gives the value of k. Subtracting this from the preceding equation, we have

$$v^2 - v'^2 = 2\left\{\phi(r') - \phi(r)\right\} \qquad (8)$$

It appears from this equation that the velocity depends solely on the distance of the moving point from the centre of force, and not at all on the nature of the path described.

5. The equations (5) and (6) are the integrals of the first order of our fundamental equations (3) and (4), which they should therefore replace; and are evidently sufficient to determine the coordinates of the moving point at each moment of time, the law of the force being known. For if the coordinates r and ω be eliminated successively from the equations

$$r^2 . \frac{d\omega}{dt} = c. \qquad \frac{dr^2}{dt^2} + r^2 . \frac{d\omega^2}{dt^2} + 2\int \mathrm{R}dr = k.$$

the resulting equations, containing each only one of the coordinates, and the time t, will, when integrated, express the relation between the time and each of those coordinates; whereby the position of the moving point will be ascertained for any given time.

Again, if the differential of the time, dt, be eliminated from those equations, the resulting equation, containing only the coordinates of the point, will give by integration the relation between r and ω, or the equation of the curve described by the moveable.

To proceed to these applications: if we substitute in the latter of those equations for $\frac{d\omega}{dt}$ its value $\frac{c}{r^2}$, obtained from the former, it becomes

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} + 2\int \mathrm{R}dr = k. \qquad (9)$$

The integral of which will give us the relation between the distance r and the time t.

In order to find the relation between ω and t, instead of eliminating r between the two differential equations above, and then integrating, we shall find it more convenient to eli-

minate that quantity between the integral of equation (9) and the equation of the orbit.

The differential equation of the orbit is found by substituting the value of dt obtained from equation (5) in (6), which thus becomes

$$c^2\left(\frac{dr^2}{r^4.d\omega^2}+\frac{1}{r^2}\right)+2\int \text{R}dr = k.$$

which equation will assume a more convenient form, if we take for the variable $u=\frac{1}{r}$; it becomes by this transformation,

$$c^2\left(\frac{du^2}{d\omega^2}+u^2\right)-2\int \text{R}.\frac{du}{u^2} = k. \qquad (10)$$

From this equation we are enabled, when the law of the force is known, or R given in function of u, to obtain by integration the relation between u and ω, or the equation of the orbit: and *vice versâ*, when the equation of the curve described is given, we may obtain by differentiation the expression for R, or the law of the variation of the force. The latter of these problems is called by English mathematicians the *direct*, and the former the *inverse problem* of central forces. We shall commence with its application to the former of these questions, as more naturally connected with our subject, reserving the latter for a subsequent part of this section.

6. The force being supposed to vary directly as the distance from the centre, let it be required to investigate the equation of the orbit.

In this case $\text{R}=mr=\frac{m}{u}$, m being the force at the unit of distance. Then, $\int \text{R}.\frac{du}{u^2}=\int\frac{mdu}{u^3}=\frac{-m}{2u^2}$, which being substituted in equation (10), we have

$$c^2\left(\frac{du^2}{d\omega^2}+u^2\right)+\frac{m}{u^2}=k.$$

the differential equation of the trajectory.

Before we proceed to integrate this equation, we shall determine the arbitrary constants by means of the maximum and minimum values of the variable u. These values are found by making $\frac{du}{d\omega}=0$ in the differential equation, which thus becomes $c^2u^2+\frac{m}{u^2}-k=0$. or,

$$u^4-\frac{k}{c^2}.u^2+\frac{m}{c^2}=0 \qquad (a)$$

and denoting the roots of this equation by u', u'', we shall have

$$\frac{k}{c^2}=u'^2+u''^2, \qquad \frac{m}{c^2}=u'^2u''^2.$$

Substituting now these values of the constants in the differential equation, and multiplying by u^2, we find

$$\frac{u^2.du^2}{d\omega^2}+u^4-(u'^2+u''^2)\,u^2+u'^2u''^2=0.$$

Now, in order to integrate this equation, let us make

$$u^2=\tfrac{1}{2}(u'^2+u''^2)-z,$$

z being the new variable. And it becomes by this transformation

$$\left(\tfrac{1}{2}\frac{dz}{d\omega}\right)^2-\left(\frac{u'^2-u''^2}{2}\right)^2+z^2=0.$$

or,

$$d\omega=\frac{\frac{1}{2}dz}{\sqrt{\left(\frac{u'^2-u''^2}{2}\right)^2-z^2}}.$$

the integral of which is

$$\omega = -\tfrac{1}{2}\,\text{arc.}\left(\cos. = \frac{z}{\tfrac{1}{2}(u'^2 - u''^2)}\right) + \alpha.$$

α being the arbitrary constant.

Now, reversing the formula, and observing that $\cos.\, 2(\alpha - \omega) = \cos.\, 2\,(\omega - \alpha)$, we obtain

$$z = \tfrac{1}{2}\,(u'^2 - u''^2).\cos.\, 2\,(\omega - \alpha).$$

whence

$$u^2 = \tfrac{1}{2}\,(u'^2 + u''^2) - \tfrac{1}{2}\,(u'^2 - u''^2).\cos.\, 2\,(\omega - \alpha). \qquad (b)$$

the complete integral.

This equation will assume a very elegant form, if we substitute for $\cos.\, 2\,(\omega - \alpha)$ its value $\cos.^2(\omega - \alpha) - \sin.^2(\omega - \alpha)$; we shall have by this substitution,

$$u^2 = u'^2.\sin.^2(\omega - \alpha) + u''^2.\cos.^2(\omega - \alpha). \qquad (c)$$

When $\omega - \alpha = 0$, $u^2 = u''^2$; and when $\omega - \alpha = \frac{\pi}{2}$, $u^2 = u'^2$. From which it appears, that u' and u'', the maximum and minimum values of u, are those which correspond to the angles $\omega = \frac{\pi}{2} + \alpha$, and $\omega = \alpha$, respectively.

Substituting for u its value $\frac{1}{r}$, we shall have

$$\frac{1}{r^2} = \frac{1}{b^2}.\sin.^2(\omega - \alpha) + \frac{1}{a^2}.\cos.^2(\omega - \alpha).$$

a and b being the greatest and least values of r; whence,

$$r^2 = \frac{a^2 b^2}{a^2.\sin.^2(\omega - \alpha) + b^2.\cos.^2(\omega - \alpha)}. \qquad (d)$$

The equation of a central ellipse, whose semiaxes are a and b.

The semiaxes a and b, may be easily expressed in terms of the constants, m, k, and c. For, solving equation (a), we shall have

$$u'^2 = \frac{\frac{1}{2}k + \sqrt{\frac{1}{4}k^2 - mc^2}}{c^2}, \qquad u''^2 = \frac{\frac{1}{2}k - \sqrt{\frac{1}{4}k^2 - mc^2}}{c^2};$$

the reciprocals of which are the values of b^2 and a^2.

7. Let us now proceed to apply equation (9) to obtain the relation between the coordinates of the curve and the time.

$$\text{R} = mr, \qquad \text{therefore, } \int \text{R}dr = m.\frac{r^2}{2}.$$

which being substituted in equation (9), it becomes

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} + mr^2 = k.$$

or multiplying by $\dfrac{r^2}{m}$,

$$\frac{1}{m}.\frac{r^2.dr^2}{dt^2} + r^4 - \frac{k}{m}.r^2 + \frac{c^2}{m} = 0.$$

The constant quantities which enter this equation have been already determined in function of u' and u'', the greatest and least values of u; for in the preceding article we have had

$$\frac{k}{c^2} = u'^2 + u''^2, \qquad \frac{m}{c^2} = u'^2u''^2.$$

whence,

$$\frac{k}{m} = \frac{u'^2 + u''^2}{u'^2u''^2} = \frac{1}{u'^2} + \frac{1}{u''^2} = a^2 + b^2.$$

$$\frac{c^2}{m} = \frac{1}{u'^2u''^2} = a^2b^2.$$

and these values being substituted in the differential equation, it is

$$\frac{1}{m}.\frac{r^2dr^2}{dt^2} + r^4 - (a^2 + b^2)r^2 + a^2b^2 = 0.$$

To integrate this equation, let us make

$$r^2 = \tfrac{1}{2}(a^2 + b^2) + z.$$

and it is transformed into

$$\frac{1}{4m} \cdot \frac{dz^2}{dt^2} + z^2 - \left(\frac{a^2-b^2}{2}\right)^2 = 0.$$

and separating the variables

$$dt = \frac{1}{2\sqrt{m}} \cdot \frac{dz}{\sqrt{\left(\frac{a^2-b^2}{2}\right)^2 - z^2}}.$$

whence,

$$t = \frac{-1}{2\sqrt{m}} \cdot \text{arc}\left(\cos. = \frac{z}{\frac{1}{2}(a^2-b^2)}\right) + \text{const.}$$

Let $t=t'$, when $z=\frac{1}{2}(a^2-b^2)$, or $r^2=a^2$; and there is, $t'=$ const. Wherefore, observing that the cosine remains the same whether the arc is positive or negative, the corrected integral is

$$t - t' = \frac{1}{2\sqrt{m}} \cdot \text{arc}\left(\cos. = \frac{z}{\frac{1}{2}(a^2-b^2)}\right). \qquad (e)$$

z being equal to $r^2 - \frac{1}{2}(a^2+b^2)$.

Reversing this formula, we have

$$z = \frac{1}{2}(a^2-b^2).\cos.2\sqrt{m}(t-t').$$

whence,

$$r^2 = \frac{1}{2}(a^2+b^2) + \frac{1}{2}(a^2-b^2)\cos.2\sqrt{m}(t-t').$$

or, substituting for $\cos.2\sqrt{m}(t-t')$ its value $\cos.^2\sqrt{m}(t-t') - \sin.^2\sqrt{m}(t-t')$,

$$r^2 = a^2.\cos.^2\sqrt{m}(t-t') + b^2.\sin.^2\sqrt{m}(t-t') \qquad (f)$$

The relation between the angle $\omega-\alpha$ and the time is readily found by substituting this value of r^2 in the equation of the orbit,

$$r^2 = \frac{a^2b^2}{a^2.\sin.^2(\omega-\alpha) + b^2.\cos.^2(\omega-\alpha)}.$$

Multiplying the result by the denominator, and reducing we find

$$a^2.\tan.^2(\omega-\alpha) - b^2.\tan.^2.\sqrt{m}(t-t') = 0.$$

whence,

$$\tan.(\omega-\alpha)=\frac{b}{a}.\tan.\sqrt{m}(t-t'). \qquad (g)$$

a remarkable expression for the relation between the angle at the centre and the time.

It appears from this equation, that the tangents of the angles $\omega-\alpha$ and $\sqrt{m}(t-t')$ vanish together, or that the angles simultaneously arrive at those values which render the tangent equal to cipher; therefore, when $\omega-\alpha=2\pi$, or the radius vector performed an entire revolution, $\sqrt{m}(t-t')=2\pi$; if, therefore, the time of an entire revolution be called T, we have

$$\mathrm{T}=\frac{2\pi}{\sqrt{m}}. \qquad (h)$$

This result, being altogether independent of a and b, the elements of the orbit, shews that all ellipses described round the same centre of force, the force varying as the distance, are described in the same time.

8. The force being supposed to vary inversely as the square of the distance, it is proposed to investigate the equation of the trajectory.

Here $\mathrm{R}=\frac{m}{r^2}=mu^2$. and $\int \mathrm{R}.\frac{du}{u^2}=\int mdu=mu$. Making this substitution in equation (10), we have

$$c^2\left(\frac{du^2}{d\omega^2}+u^2\right)-2mu=k.$$

the differential equation of the curve.

Proceeding as in Art. 6. let u' and u'' be the roots of the quadratic equation,

$$u^2-\frac{2m}{c^2}.u-\frac{k}{c^2}=0. \qquad (i)$$

which is obtained by making $\frac{du}{d\omega}=0$. in the differential equation; we have

$$\frac{2m}{c^2}=u'+u'', \qquad \frac{-k}{c^2}=u'u''.$$

and these values of the constants being substituted in the differential equation, it becomes

$$\frac{du^2}{d\omega^2}+u^2-(u'+u'')u+u'u''=0.$$

In order to integrate this equation, let us make

$$u=\tfrac{1}{2}(u'+u'')+z.$$

and it becomes by this transformation

$$\frac{dz^2}{d\omega^2}+z^2-\left(\frac{u'-u''}{2}\right)^2=0. \qquad \text{or, } d\omega=\frac{dz}{\sqrt{\left(\frac{u-u''}{2}\right)^2-z^2}}.$$

and integrating,

$$\omega=-\text{arc}\left(\cos.=\frac{z}{\frac{1}{2}(u'-u'')}\right)+\alpha.$$

or reversing the formula,

$$z=\tfrac{1}{2}(u'-u'').\cos.(\omega-\alpha).$$

Therefore,

$$u=\tfrac{1}{2}(u'+u'')+\tfrac{1}{2}(u'-u'').\cos.(\omega-\alpha). \qquad (k)$$

the equation of the curve.

This equation may be put under a form similar to that of equation (c), Art. 6., and by precisely the same transformation. Substitute for cos. $(\omega-\alpha)$ its value $\cos^2\frac{\omega-\alpha}{2}-\sin^2\frac{\omega-\alpha}{2}$, and we shall have

$$u=u'.\cos^2\frac{\omega-\alpha}{2}+u''.\sin^2\frac{\omega-\alpha}{2}. \qquad (l)$$

When $\omega-\alpha=0$, $u=u'$; and when $\omega-\alpha=\pi$, $u=u''$. It appears, therefore, that u' and u'', the maximum and minimum values of u, are those which correspond to the angles

$\omega=\alpha$, and $\omega=\pi+\alpha$. They are, therefore, in the same right line.

If we substitute for u, u', and u'', their values $\frac{1}{r}$, $\frac{1}{r'}$, and $\frac{1}{r''}$ in equation (k), it becomes

$$\frac{1}{r}=\tfrac{1}{2}\left(\frac{1}{r'}+\frac{1}{r''}\right)+\tfrac{1}{2}\left(\frac{1}{r'}-\frac{1}{r''}\right)\cos.(\omega-\alpha).$$

or,

$$r=\frac{r'r''}{\frac{1}{2}(r''+r')+\frac{1}{2}(r''-r').\cos.(\omega-\alpha)}.$$

And making $\frac{1}{2}(r''+r')=a$, $\frac{1}{2}(r''-r')=ae$, there shall be $r'r''=a^2-a^2e^2$; and the equation becomes

$$r=\frac{a(1-e^2)}{1+e.\cos.(\omega-\alpha)}. \qquad (m)$$

the known equation of a conic section referred to the focus as the origin.—(See *Analytic Geometry*, Art. 51.)

9. For the determination of the constants a and e, we have

$$a=\tfrac{1}{2}(r'+r'')=\frac{\frac{1}{2}(u'+u'')}{u'u''},$$

and

$$ae=\tfrac{1}{2}(r''-r')=\frac{\frac{1}{2}(u'-u'')}{u'u''}, \text{ whence } e=\frac{u'-u''}{u'+u''}.$$

But we have had above, the equations

$$\tfrac{1}{2}(u'+u'')=\frac{m}{c^2}, \qquad u'u''=-\frac{k}{c^2}.$$

from which we deduce $\frac{1}{2}(u'-u'')=\frac{\sqrt{m^2+kc^2}}{c^2}$; substituting these values in the expressions for a and e, we find

$$a=-\frac{m}{k}, \qquad e=\sqrt{1+k.\frac{c^2}{m^2}}. \qquad (n)$$

It is a problem, however, of more importance to determine the elements of the orbit a and e by means of the initial velocity, distance, and angle of projection; or, in other words, a body being projected from a given point, in a given direction, and with a given velocity, to determine the particular conic section, which it will describe. This is done by finding the values of c and k in terms of the velocity, distance, and angle of projection, and substituting them in the equations (n).

If θ be the angle formed by the tangent with the radius vector, $\sin.\theta = \frac{rd\omega}{ds}$, ds being the element of the curve. Multiplying by r, $r.\sin.\theta = \frac{r^2d\omega}{ds} = \frac{cdt}{ds}$, by equation (5).

$$\therefore\ c = \frac{ds}{dt}.r.\sin.\theta = v.r.\sin.\theta.$$

Again, by equation (7), $v^2 = k - 2\int \mathrm{R}dr = k + \frac{2m}{r}$, when the force varies inversely as the square of the distance, or $\mathrm{R} = \frac{m}{r^2}$. Therefore, if ρ be the distance of the point from the centre at the commencement of the motion, V the velocity, and ε the angle of projection, we shall have

$$c = \mathrm{V}.\rho.\sin.\varepsilon, \qquad k = \mathrm{V}^2 - \frac{2m}{\rho}. \qquad (o)$$

and substituting these values in equation (n),

$$a = \frac{m}{\frac{2m}{\rho} - \mathrm{V}^2}, \qquad e^2 = 1 + \frac{\mathrm{V}^2\rho^2.\sin.^2\varepsilon\left(\mathrm{V}^2 - \frac{2m}{\rho}\right)}{m^2}. \qquad (p)$$

The value of a thus found, is independent of the angle of projection; from which it appears, that in whatever direction the body is projected, the axis major of the section will be the same, if the velocity remains unaltered.

The curve described shall be an ellipse, hyperbola, or parabola, as the excentricity e is less, greater than, or equal to unity: that is, according as

$$1-e^2=\frac{\text{v}^2\rho^2.\sin.^2\varepsilon\left(\frac{2m}{\rho}-\text{v}^2\right)}{m^2}$$

is positive, negative, or cipher. But the sign of $1-e^2$ is evidently the same as that of $\frac{2m}{\rho}-\text{v}^2$. Therefore, the curve shall be an ellipse, hyperbola, or parabola, according as v^2 is less, greater than, or equal to $\frac{2m}{\rho}$.

10. Let us now proceed to investigate the time of describing any portion of the orbit.

Since $\text{R}=\frac{m}{r^2}$, $\int \text{R}dr=\int m.\frac{dr}{r^2}=\frac{-m}{r}$. If then this value is substituted in the equation,

$$\frac{dr^2}{dt^2}+\frac{c^2}{r^2}+2\int \text{R}dr=k.$$

and the result multiplied by r^2, there is

$$\frac{r^2dr^2}{dt^2}-kr^2-2mr+c^2=0.$$

The values of the constants k and c^2 have been already obtained in function of the elements of the orbit: for from equations (n), Art. 9, we have

$$k=-\frac{m}{a}, \qquad c^2=ma(1-e^2).$$

If now we substitute these values in our differential equation, and multiply the result by $\frac{a}{m}$, it becomes

$$\frac{a}{m}.\frac{r^2dr^2}{dt^2}+(a-r)^2-a^2e^2=0.$$

or, separating the variables,

$$dt = \frac{\sqrt{\frac{a}{m}}.rdr}{\sqrt{a^2e^2-(a-r)^2}}$$

In order to integrate this expression, let us make $a-r = ae.z$, whence we have

$$r = a(1-ez), \qquad dr = -ae.dz.$$

By this substitution, the equation becomes

$$dt = -\frac{a\sqrt{a}}{\sqrt{m}}. \frac{(1-ez)dz}{\sqrt{1-z^2}}.$$

the integral of which is

$$t = \frac{a\sqrt{a}}{\sqrt{m}} \{\text{arc.}(\cos. = z) - e.\sqrt{1-z^2}\} + \text{const.}$$

If t' be the value of t, when $z=1$, or $r = a(1-e)$, we shall have $t' = $ const.; and the corrected integral is

$$t - t' = \frac{a\sqrt{a}}{\sqrt{m}} \{\text{arc.}(\cos. = z) - e\sqrt{1-z^2}\} \qquad (q)$$

t' being the time elapsed when the body arrives at the lower apsis.

$t-t'$ being the time counted from the lower apsis, the time of a semi-revolution will be had by making $r = a(1+e)$, the higher apsidal distance, or $z = -1$; which gives $\frac{a\sqrt{a}}{\sqrt{m}}.\pi$ for the time of half a revolution. If, therefore, the time of an entire revolution be called T, we have

$$\text{T} = 2\pi.\frac{a\sqrt{a}}{\sqrt{m}}. \qquad (r)$$

From which it appears, that the periodic times in different orbits vary in the sesquiplicate ratio of the major axes; this is the third law of the planetary movements discovered by Kepler.

The relation between t and z or r, given by equation (q),

is generally exhibited under the form of two equations, which are virtually contained in that equation, and flow from it by introducing a new variable, namely, the arc whose cosine $= z$: for if this be called ϕ, there shall be

$$z = \cos.\phi, \quad t - t' = \frac{a\sqrt{a}}{\sqrt{m}}(\phi - e.\sin.\phi).$$

or making for conciseness $\dfrac{\sqrt{m}}{a\sqrt{a}} = n$, and observing that $r = a(1 - ez)$,

$$r = a(1 - e.\cos.\phi). \qquad (s)$$

$$n(t - t') = \phi - e.\sin.\phi. \qquad (t)$$

By the latter of these equations ϕ may be expressed by means of $t - t'$, and this substituted in the former will give the radius vector, r, in function of the time, or conversely.

It remains now to discover the relation between ω and t. For this purpose we must eliminate r between equation (s) and the equation of the orbit: we thus have

$$1 - e.\cos.\phi = \frac{1 - e^2}{1 + e.\cos.(\omega - \alpha)}.$$

whence we find

$$\cos.(\omega - \alpha) = \frac{\cos.\phi - e}{1 - e.\cos.\phi}.$$

from this we obtain the values of $1 - \cos.(\omega - \alpha)$ and $1 + \cos.(\omega - \alpha)$, which, divided one by the other, give

$$\frac{1 - \cos.(\omega - \alpha)}{1 + \cos.(\omega - \alpha)} = \frac{1 + e}{1 - e}.\frac{1 - \cos.\phi}{1 + \cos.\phi}.$$

or,

$$\tan.\frac{\omega - \alpha}{2} = \sqrt{\frac{1 + e}{1 - e}}\tan.\frac{\phi}{2}. \qquad (v)$$

If, therefore, ϕ be found in terms of $t - t'$, by means of equation (t), and the result substituted in (v), we shall have

$\omega-\alpha$ expressed by means of $t-t'$, or the angle in function of the time.

In order to represent the planetary movements, astronomers have supposed a star, which sets out from perihelion at the same time with the planet, and moves with a uniform angular velocity round the sun, the angle $n\,(t-t')$ formed by the radius vector of this star with the axis major, at any time t, is called by them the *mean anomaly;* the angle $\omega-\alpha$ formed by the radius vector of the planet, the *true anomaly;* and the angle ϕ the *excentric anomaly.*

11. Let the expression for the force consist of two parts, one of which varies inversely as the square, and the other inversely as the cube of the distance; and let it be required to find the equation of the curve described by the body.

$$\mathrm{R} = mu^2 + m'u^3,$$

$$\therefore \int \mathrm{R}.\frac{du}{u^2} = \int (m + m'u)\, du = mu + m'\frac{u^2}{2}.$$

making this substitution in equation (10), and arranging, we have for the differential equation of the orbit

$$c^2.\frac{du^2}{d\omega^2} + (c^2 - m')\, u^2 - 2mu - k = 0.$$

If we make $\frac{du}{d\omega} = 0$ in this equation, the roots of the resulting equation will be the greatest and least values of u. Therefore, u' and u'' being those values, or the roots of the equation,

$$u^2 - \frac{2m}{c^2 - m'}.\,u - \frac{k}{c^2 - m'} = 0. \qquad (w)$$

we shall have

$$\frac{2m}{c^2 - m'} = u' + u'', \qquad \frac{-k}{c^2 - m'} = u'u''.$$

Therefore, dividing the differential equation by $c^2 - m'$, sub-

stituting these values in the result, and making $\frac{c^2-m'}{c^2}=\gamma^2$, it becomes

$$\frac{1}{\gamma^2}.\frac{du^2}{d\omega^2}+u^2-(u'+u'')u+u'u''=0.$$

In order to integrate, let $\frac{1}{2}(u'+u'')+z$ be substituted for u, and the equation is transformed into

$$\frac{1}{\gamma^2}.\frac{dz^2}{d\omega^2}+z^2-\left(\frac{u'-u''}{2}\right)^2=0.$$

or, separating the variables,

$$d\omega=\frac{1}{\gamma}\,\frac{dz}{\sqrt{\left(\frac{u'-u''}{2}\right)^2-z^2}}.$$

whence,

$$\omega=-\frac{1}{\gamma}.\text{arc.}\left\{\cos.=\frac{z}{\frac{1}{2}(u'-u'')}\right\}+\alpha.$$

or, reversing the formula,

$$z=\tfrac{1}{2}(u'-u'').\cos.\gamma(\omega-\alpha),$$

$$\therefore\ u=\tfrac{1}{2}(u'+u'')+\tfrac{1}{2}(u'-u'').\cos.\gamma(\omega-\alpha). \qquad (x)$$

If we substitute in this equation for $\cos.\gamma(\omega-\alpha)$ its value $\cos.^2\gamma\left(\frac{\omega-\alpha}{2}\right)-\sin.^2\gamma\left(\frac{\omega-\alpha}{2}\right)$, it will assume the form,

$$u=u'.\cos.^2\gamma\left(\frac{\omega-\alpha}{2}\right)+u''.\sin.^2\gamma\left(\frac{\omega-\alpha}{2}\right) \qquad (y)$$

An equation which coincides with that of the focal ellipse, except in this, that the variable $(\omega-\alpha)$ is multiplied by a constant quantity. Hence this curve may be readily constructed; for, in the ellipse whose apsidal distances are $\frac{1}{u'}$ and $\frac{1}{u''}$, if a line be drawn from the focus forming an angle with the axis, which is to the angle formed by the radius vector of the ellipse with the same, in the ratio of 1 to

γ, and this line be taken equal to that radius vector, its extremity shall describe the curve in question.

When $\omega-\alpha=0$, in equation (y), we shall have $u=u'$; and when $\gamma(\omega-\alpha)=\pi$, there is $u=u''$. Therefore, ω' and ω'' being the values of ω which correspond to u' and u'', the greatest and least values of u, we shall have

$$\omega'=\alpha, \qquad \omega''=\alpha+\frac{\pi}{\gamma}.$$

Therefore, subtracting, the angle between the apsides,

$$\omega''-\omega'=\frac{\pi}{\gamma}=\frac{\pi}{\sqrt{1-\frac{m'}{c^2}}} \qquad (z)$$

12. The force being represented by any function of the distance; it is required to find the equation of the orbit, when it approaches very nearly to a circle.

The force R being a function of the distance from the centre, or its reciprocal u; $\frac{\text{R}}{u^2}$ shall be a function of the same quantity, which we shall denote by the character $f(u)$; whence $\text{R}=u^2.f(u)$. Also, $\int \text{R}.\frac{du}{u^2}=\int f(u).du$; denoting this integral by the character $f_{\prime}(n)$, and substituting in equation (10), we have

$$c^2\left(\frac{du^2}{d\omega^2}+u^2\right)-2f_{\prime}(u)=k.$$

Now it is evident that if the body is projected in a direction perpendicular to the radius vector, and with a velocity very nearly equal to that in a circle at the same distance, the orbit will not differ much from a circle. But by equation (5) we have

$$c=r^2.\frac{d\omega}{dt}=r^2.\frac{ds}{dt}.\frac{d\omega}{ds}=rv.\sin.\theta.$$

Hence, a being the value of u, or the reciprocal of the dis-

tance r, at the place of projection, and V the velocity of projection, we have

$$c = \frac{\mathrm{V}}{a}.$$

the angle of projection being right, and therefore $\sin.\theta = 1$. Again, by equation (7),

$$v^2 = k - 2\int \mathrm{R}dr = k + 2\int \mathrm{R}.\frac{du}{u^2} = k + 2f_{\prime}(u).$$

But when $u = a$, $v = \mathrm{V}$; wherefore,

$$k = \mathrm{V}^2 - 2f_{\prime}(a).$$

and these values of c and k being substituted in the differential equation, it becomes

$$\frac{\mathrm{V}^2}{a^2}\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\ \{f_{\prime}(u) - f_{\prime}(a)\} = \mathrm{V}^2.$$

Now let $u = a + z$, a being the value of u at the point of projection; and z a very small but variable quantity; then, $du = dz$, and by Taylor's theorem,

$$f_{\prime}(a+z) - f_{\prime}(a) = f_{\prime}(a).\frac{z}{1} + f'(a).\frac{z^2}{1.2} + f''(a).\frac{z^3}{1.2.3} + \&c.$$

f being the first derived function of $f_{\prime}$.

Therefore, making these substitutions in the equation above, and dividing the whole by $\frac{\mathrm{V}^2}{a^2}$, we find

$$\frac{dz^2}{d\omega^2} + 2az + z^2 - \frac{2a^2}{\mathrm{V}^2}\left(f(a).\frac{z}{1} + f'(a).\frac{z^2}{1.2} + \&c.\right) = 0.$$

Putting V′ for the velocity in a circle at the point of projection, there is $\mathrm{V}'^2 = \mathrm{R} \times$ radius (Equat. (f'), Art. 14.) $= a^2.f(a).\frac{1}{a} = a.f(a)$. But by the conditions of the problem, $\mathrm{V}^2 : \mathrm{V}'^2 :: 1 : 1 + \delta$, δ being a very small quantity; therefore,

$$\mathrm{V}^2 = \frac{\mathrm{V}'^2}{1+\delta} = \frac{a.f(a)}{1+\delta}, \qquad \text{and } \frac{a^2}{\mathrm{V}^2} = \frac{a(1+\delta)}{f(a)}.$$

substituting now this value in the equation just obtained, neglecting the terms which contain z^3, $z^2\delta$, &c. as indefinitely small compared to the rest, and arranging, we obtain

$$\frac{dz^2}{d\omega^2}+\left(1-\frac{a.f'(a)}{f(a)}\right)z^2-2a\delta.z=0.$$

the differential equation of the curve.

In order to integrate this equation, let us make $1-\frac{a.f'(a)}{f(a)}=\gamma^2$, and dividing the equation by γ^2, it will assume the form

$$\frac{1}{\gamma^2}.\frac{dz^2}{d\omega^2}+z\left(z-\frac{2a\delta}{\gamma^2}\right)=0.$$

or, putting for z its equal $u-a$.

$$\frac{1}{\gamma^2}.\frac{du^2}{d\omega^2}+(u-a)\left(u-a-\frac{2a\delta}{\gamma^2}\right)=0.$$

If we make $\frac{du}{d\omega}=0$ in this equation, the roots of the resulting equation will be the values of u at the apsides, or the greatest and least values of u; we thus find

$$u=a, \qquad u=a\left(1+\frac{2\delta}{\gamma^2}\right). \qquad (a')$$

These values being, as before, denoted by u'' and u', the above equation will become

$$\frac{1}{\gamma^2}.\frac{du^2}{d\omega^2}+u^2-(u'+u'')u+u'u''=0.$$

the same differential equation which we have had in the last article, whose integral is

$$u=\tfrac{1}{2}(u'+u'')+\tfrac{1}{2}(u'-u'')\cos.\gamma(\omega-\alpha).$$

or,

$$u=u'.\cos.^2\gamma\left(\frac{\omega-\alpha}{2}\right)+u''.\sin.^2\gamma\left(\frac{\omega-\alpha}{2}\right) \qquad (b')$$

and if we substitute in the latter of these equations for u'

and u'' their values $a\left(1+\frac{2\delta}{\gamma^2}\right)$ and a, we shall have the equation of the curve,

$$u=a\left\{1+\frac{2\delta}{\gamma^2}.\cos.^2\gamma\left(\frac{\omega-\alpha}{2}\right)\right\} \qquad (c')$$

13. It appears from Art. 11. that the angle between the apsides in this curve is

$$\omega''-\omega'=\frac{\pi}{\gamma}=\frac{\pi}{\sqrt{1-\frac{a.f'(a)}{f(a)}}} \qquad (d')$$

When the force varies as any power of the distance, or $R=mu^n$, $f(u)=\frac{R}{u^2}=mu^{n-2}$, and $f'(u)=m(n-2)u^{n-3}$; therefore,

$$1-\frac{a.f'(a)}{f(a)}=1-\frac{m(n-2)a^{n-2}}{m.a^{n-2}}=3-n.$$

and the angle between the apsides $=\frac{\pi}{\sqrt{3-n}}$.

Hence, when the force varies directly as the distance, or $n=-1$, the angle between the apsides $=\frac{\pi}{2}$, as we have already seen, Art. 6. When the force varies inversely as the square of the distance, or $n=2$, the angle between the apsides $=\pi$, as has been already shown, Art. 8. When the force varies inversely as the cube of the distance, the angle between the apsides is infinite: and the force varying inversely as any higher power of the distance, the value of the angle becomes imaginary; which indicates that the body leaving an apsis will never arrive at another; and, therefore, that it shall either descend to the centre or go off to infinity.

Let the force consist of two parts, each varying as some power of the distance, or $R=mu^n+m'u^{n'}$. In this case,

$$f(u)=\frac{R}{u^2}=mu^{n-2}+m'u^{n'-2}.$$

and

$$f'(u) = m(n-2)u^{n-3} + m'(n'-2)u^{n'-3}.$$

$$\therefore 1 - \frac{a.f'(a)}{f(a)} = 1 - \frac{(n-2)ma^{n-2} + (n'-2)m'a^{n'-2}}{ma^{n-2} + m'a^{n'-2}} =$$

$$\frac{(3-n)ma^{n-2} + (3-n')m'a^{n'-2}}{ma^{n-2} + m'a^{n'-2}}.$$

and the angle between the apsides is equal to

$$\pi . \sqrt{\frac{ma^{n-2} + m'a^{n'-2}}{(3-n)ma^{n-2} + (3-n')m'a^{n'-2}}}.$$

14. In all that has preceded, we have supposed the forces given, and thence determined the nature and circumstances of the motion. We shall now proceed in an inverse method, by supposing the laws of the movement known, and thence deducing the nature of the forces, and the laws according to which they vary.

This method may be applied to the motion of the heavenly bodies, which, as it will afford the most convenient illustration, will moreover lead to the most important results.

Let us suppose, therefore, in the first place, as was judged to be the case in the infancy of astronomical knowledge, that the planets move in circular orbits round the sun, and with an equable motion. In this case, it is evident, the areas described round the sun are proportional to the times of their description, and therefore it follows, from Art. 6. and 7. of the preceding section, that the forces are always directed to the sun. In investigating the nature of these forces, therefore, we may employ the theorems respecting central forces obtained in the preceding part of this section.

For the case under consideration it will be most convenient to make use of equation (4), which, since r is constant, becomes

$$\text{R} = r . \frac{d\omega^2}{dt^2}.$$

Now $\frac{d\omega}{dt}$ expresses the angular velocity, and as this by hypothesis is constant, we have $\frac{d\omega}{dt} = \frac{2\pi}{\text{T}}$, T being the time of the entire revolution; therefore, making this substitution

$$\text{R} = 4\pi^2 . \frac{r}{\text{T}^2} \qquad (e')$$

the general expression for the force in a circle.

This formula, however, will not enable us to determine the variation of the solar force dependent on the change of distance, unless it should appear that there existed some relation between the periodic times and the distances. Here observation comes to our aid, by which it appears that the squares of the periodic times of the planets vary as the cubes of their distances from the sun; and from this result, combined with the expression for the force already found, we conclude that the forces by which the different planets are urged towards the sun vary inversely as the squares of their distances from that body.

V being the velocity, we have $\text{V} = \frac{r d\omega}{dt}$, therefore,

$\text{V}^2 = r^2 . \frac{d\omega^2}{dt^2} = \text{R}r$, or

$$\text{V} = \sqrt{\text{R}.r}. \qquad (f')$$

This is the general expression for the velocity in a circle described under the influence of a force directed to the centre. But, the force varying inversely as the square of the distance, as has been proved to be the case in the planetary system, V varies as $\frac{1}{\sqrt{r}}$; *i. e.* the velocities of the different planets vary inversely in the subduplicate ratio of the distances.

15. As the science of astronomy advanced, it was found that the circular hypothesis did not exactly accord with ob-

servation; the true laws of the planetary movements were first discovered by Kepler, and are as follows:

1st, Each planet describes equal areas in equal times about the sun.

2d, The orbits described by the planets are ellipses, the sun being in one of the foci.

3d, The periodic times of the different planets are in the sesquiplicate ratio of the major axes of their orbits.

From the first of these laws we conclude, as before, that the force, by which each of the planets is urged, is constantly directed to the sun; thus the nature of the force is determined.

The area described in the time dt being $\frac{1}{2}r^2d\omega$, we have by this law

$$r^2d\omega = cdt.$$

Whence, the orbit being given, we can find the time of describing any portion of it, and also the velocity at any point. For, in the first place, we have $dt = \frac{r^2d\omega}{c}$, in which, by means of the equation of the orbit, ω may be expressed in terms of r, or *vice versa*, and the time then found by integrating.

Again, the expression of the velocity is $v = \frac{ds}{dt}$, ds being the differential of the arc of the curve; but $ds^2 = dr^2 + r^2d\omega^2$, and $dt^2 = \frac{r^4d\omega^2}{c^2}$; whence $v^2 = c^2\left(\frac{dr^2}{r^4d\omega^2} + \frac{1}{r^2}\right)$, or, substituting $\frac{1}{u}$ for r.

$$v^2 = c^2\left(\frac{du^2}{d\omega^2} + u^2\right). \qquad (11)$$

It is easily shown that $\frac{du^2}{d\omega^2} + u^2 = \frac{1}{p^2}$, p being the perpendicular from the origin on the tangent; wherefore,

$v = \frac{c}{p}$; or the velocity at any point varies inversely as the perpendicular let fall from the centre of force on the tangent to the curve at that point; as may be shown very readily from geometric considerations.

16. The relation between the coordinates of the moving body, and the central force by which it is urged, is given by equation (10). That equation has been hitherto applied to discover the relation between those coordinates, or the equation of the curve described by the body, when the force is given. In the present case, however, the path of the body is given, and it is required to deduce the law of the force. For this purpose let equation (10) be differentiated, and the result divided by $2du$; it becomes

$$c^2\left(\frac{d^2u}{d\omega^2} + u\right) - \frac{\text{R}}{u^2} = 0.$$

whence there is

$$\text{R} = c^2u^2\left(\frac{d^2u}{d\omega^2} + u\right) \tag{12}$$

the general expression for the central force in any curve described by its action combined with an initial projection.

Now by the second of Kepler's laws, the curve described by each of the planets is an ellipse, the sun being in the focus. The polar equation of the ellipse referred to the focus is

$$u = \frac{1 + e.\cos.(\omega - \alpha)}{a(1-e^2)}.$$

and differentiating twice, we have

$$\frac{d^2u}{d\omega^2} = \frac{-e.\cos.(\omega-\alpha)}{a(1-e^2)}, \qquad \text{whence } \frac{d^2u}{d\omega^2} + u = \frac{1}{a(1-e^2)};$$

and this being substituted in the general expression for the force, we find

$$\text{R} = \frac{c^2u^2}{a(1-e^2)} = \frac{m}{r^2}, \tag{g'}$$

making $\frac{c^2}{a(1-e^2)}=m$. The force, therefore, by which each planet is solicited, varies inversely as the square of its distance from the sun.

17. The quantity m expresses the intensity of this force at the unit of distance; this is constant for any one planet, but we do not as yet know whether it may change or not in passing from one planet to another. The third law of Kepler will enable us to determine this point.

Since the areas are proportional to the times of describing them, and c expresses the double of the area described in the unit of time, $c\text{T}$ will be double of the area described in the time T; that is, double the area of the ellipse, T being the time of an entire revolution. Now the area of the ellipse $=\pi ab$. (a and b being the semi-axes,) $=\pi a^2\sqrt{1-e^2}$; wherefore, $c=2\pi.\frac{a^2\sqrt{1-e^2}}{\text{T}}$, and

$$m=\frac{c^2}{a(1-e^2)}=4\pi^2.\frac{a^3}{\text{T}^2} \qquad (h')$$

Now m' being the absolute force for any other planet, whose semi-axis major is a', and periodic time T' there is

$$m'=4\pi^2.\frac{a'^3}{\text{T}'^2}.$$

But by the third law of Kepler, $\frac{a^3}{\text{T}^2}=\frac{a'^3}{\text{T}'^2}$; wherefore, $m=m'$, or the absolute force is the same for all the planets. If, therefore, we suppose the planets placed at equal distances from the sun, they shall all be urged by the same accelerating force; and being let fall without any initial velocity, they would descend towards the sun, describing equal spaces in the same time.

Thus by Kepler's laws, which are the results of observation, Newton was led to the discovery of the nature of those forces which animate the planetary system. From the

proportionality of the areas to the times of their description, it has been shown that the force soliciting each planet is directed always to the centre of the sun. From the ellipticity of the planetary orbits, we have concluded, that, for each planet, this force varies inversely as the square of the distance from the sun. And, finally, from the third law of Kepler, which has been called the *harmonic law,* it follows, that this attractive force is the same for all the planets, if placed at equal distances from the sun; and, therefore, that it does not change from one planet to another, but in consequence of the change of distance.

SECTION V.

OF CONSTRAINED MOTION IN GENERAL.

1. If a material point or body, subjected to the action of a certain force, p, is constrained by a rigid line or surface to take a direction different from that of p, this force naturally resolves itself into two forces; one of them perpendicular, and the other parallel to the line or surface. This latter, called the tangential force, shall produce its full effect in accelerating or retarding the motion of the body: but the normal force is equilibrated by the reaction. This normal force constitutes the pressure immediately arising from the force p; and being independent of the movement, it is the only pressure we have had occasion to consider in unfolding the theory of Statics.

If several forces, p, p', p'', &c. act on the body, what has been stated relative to a single force is to be understood of their resultant.

But that this is not the whole amount of the pressure made by the body, moving on a curved surface, may appear by supposing the action of the forces p, p', p'', &c. to have ceased. For the continual deflection of the body from the course it would have taken if unrestrained, evinces the existence of a force of reaction, and, therefore, of a pressure against the constraining line or surface.

The better to conceive the nature of this deflecting force, the figure, to which the body is constrained to accommodate its motion, may be supposed to be a polygon. Let $mm'm''$, &c. be that polygon, (Fig. 143.), and ω the angle made by the side $m'm''$, with the side mm' produced beyond the point

m'. When the body has described the line mm' with a certain velocity v, instead of continuing its course in that line produced, it is constrained to move in the contiguous side $m'm''$. The velocity with which it moves in the line $m'm''$, is had by resolving v according to the line $m'm''$, and the perpendicular to that line. The velocities in these directions are $v.\cos.\omega$, and $v.\sin.\omega$, respectively. The former of these is the velocity retained in the side $m'm''$, and the latter is that destroyed by its reaction.

2. If the constraining line or surface is curved, the change of direction is continuous; and, therefore, the reaction by which this change is produced, is unremitting. The angle ω, which measures the deflection in passing from one element to another, is indeed indefinitely small; but then, the time dt is also indefinitely small: and when the latter becomes a definite quantity, the angle which measures the change of direction also becomes of a definite magnitude. The force of reaction, in this case, being unremitting, and moreover, requiring some definite time to produce a definite effect, is comparable to an accelerating or retarding force. Its value is expressed by the velocity destroyed in an indefinitely small time divided by that time, *i.e.* by $\frac{v.\sin.\omega}{dt}$, or $\frac{v.\omega}{dt}$; the arc, when indefinitely small, being confounded with its sine.

To find a more convenient expression, let mm' be the element of the curve described by the body, in the time dt, (Fig. 144.), and let two tangents be drawn to the points m, m', intersecting at o, the former of these being produced, as to n. The deflection or change of direction, which occurs in the passage from m to m', is then measured by the angle nom'. Now if mc,$m'c$ are perpendiculars to the tangents at the points m and m', meeting at c, the angle nom' is equal to the angle at c, which, therefore, may be denoted by ω. Also, putting ds for the indefinitely small arc mm',

and ρ for mc or $m'c$, which is then the radius of curvature, we have $\omega = \frac{ds}{\rho} = \frac{v.dt}{\rho}$. Wherefore, substituting this value of ω, the deflecting force $\frac{v.\omega}{dt} = \frac{v^2}{\rho}$. The same is the measure of the pressure arising from the tendency of the body to move in a right line: a tendency which is counteracted, but not destroyed by the normal forces. The force $\frac{v^2}{\rho}$ being always directed from the centre of the circle of curvature, is denominated the centrifugal force.

3. If a body is constrained to move in a circle, the radius of curvature is the radius of the circle actually described; and, r denoting that radius, and f the centrifugal force, this force is expressed by the equation $f = \frac{v^2}{r} = \frac{2gh}{r}$; in which h denotes the height due to v, the velocity of the gyrating body. This equation immediately gives the relation of the centrifugal force in a circle to the force of gravity; for dividing both sides of the equation by g, there is

$$\frac{f}{g} = \frac{2h}{r}.$$

The equation $f = \frac{v^2}{r}$ may be changed for another, expressing the centrifugal force by the time of the revolution: for, denoting this time by T, there is $v = \frac{2\pi r}{T}$; which value being substituted, the equation becomes

$$f = \frac{4\pi^2 r}{T^2}.$$

The time T is found by dividing any certain portion of time by the number of gyrations made in that time: and the facility with which T is thus obtained, even in the case of rapid movements, renders this expression particularly convenient, whenever it is proposed to find the absolute value of the centrifugal force. For example; for the centrifugal

force at the equator, produced by the rotation of the earth round its axis, we have $\pi = 3.1415926$, $T = 86164$, the number of seconds in a sidereal day; and for the equatorial radius $r = 20,976876$ feet. Making these substitutions in the preceding formula, there is $f = .11154$.

The centrifugal force at the equator is, therefore, such as would generate a velocity of .11154 feet, in a second of time. To compare this with the force of gravity at the equator, the latter, estimated in the same way, is 32.125, consequently,

$$f = \frac{g}{288}.$$

i. e. the centrifugal force at the equator is the 288th part of the effective force of gravity. But if G denote the attraction of the terrestrial spheroid

$$g = G - f;$$

so that we have

$$f = \frac{G}{289}.$$

or the centrifugal force at the equator is the 289th part of the total gravity.

Were the time of the earth's revolution diminished in any ratio, the centrifugal force would be increased in the duplicate of that ratio. It is, therefore, easy to find the time in which it should revolve, in order that the centrifugal force at the equator should be equal to that of the attraction towards the spheroid. Since, putting T' for this time, we have $T'^2 : T^2 :: 1 : 289$. Wherefore, $T' = \frac{T}{17}$, showing that if the earth were to revolve round its axis in the 17th part of a sidereal day, or in $1^h, 24^m, 28\frac{1}{2}''$, the force of gravity at the equator would be equilibrated by the centrifugal force; and the bodies on this part of the earth's surface would have no

weight. In this statement it is supposed that the figure of the earth remains unchanged.

When a solid revolves round an axis, each point describes a circle whose radius is the perpendicular let fall from that point on the axis: and all those circles being described in the same time, the centrifugal forces of the several points are proportional to their perpendicular distances from the axis. Consequently, if f' denote the centrifugal force at any parallel of latitude, and r' the radius of the parallel,

$$f' = f \cdot \frac{r'}{r} = f \cos. l;$$

l being the latitude of the parallel. This force is directed from the centre of the circle of latitude. The portion of it which acts in opposition to gravity, is found by resolving it in the direction of the terrestrial radius, or by multiplying it by the cosine of latitude; so that the diminution of gravity arising from the centrifugal force is

$$f' \cos. l = f \cos.^2 l.$$

To show this, let PEP′E′ be a great circle passing through the poles P, P′; and let EC be the equatorial radius; BD that of any parallel, (Fig. 145.) Then, if CE represents the centrifugal force at E, that at B shall be represented by DB; and DM being a perpendicular from D, on the radius CB, the part opposed to gravity shall be MB. But CB : DB : MB, *i. e.* CE : DB : MB; wherefore, CE : MB : : CE^2 : DB^2, or as $1 : \cos.^2 l$. Whence it follows, that the decrement of gravity, immediately due to the centrifugal force, varies as the square of the cosine of latitude.

4. Many of the laws of unconstrained motion flow readily from the principles here laid down. For when the body is free, the forces acting on it must supply the place of the reaction of the curve; so that if the resultant of these forces be resolved into two, one in the direction of the tangent, and the other perpendicular to it; the latter must be equal and

opposite to the centrifugal force. Hence it follows, that the osculating plane of the curve, at any point, must contain the direction of that resultant; and if θ be the angle which that direction makes with the tangent to the curve, we must have

$$\text{R}.\sin.\theta = \frac{v^2}{\rho}.$$

When the resultant, R, is directed to a fixed centre, the curve will all lie in one plane. In this case the angle θ is deduced from the equation of the curve itself, and the law of the central force is given by the formula

$$\text{R} = \frac{v^2}{\rho\sin.\theta}.$$

If 2γ denote the chord of curvature passing through the centre of force, $\gamma = \rho\sin.\theta$; and we have the formula of Newton,

$$\text{R} = \frac{v^2}{\gamma}.$$

When the body moves in a circle, whose centre is the centre of force, $\gamma = r$, and

$$\text{R} = \frac{v^2}{r}.$$

5. To return to the problem of constrained motion. Ths forces by which the movement is affected are the forces p, p', p'', &c. or their resultant, R, and the reaction of the line or surface. The force R is resolved into two, viz. a normal and tangential force. The reaction is likewise resolvable into two forces, both of them normal. Of these, one is equal and opposite to the normal force exerted by R, and the other that expressed by $\frac{v^2}{\rho}$. The equal and opposite forces being suppressed, there are only the force R resolved according to the tangent, and that part of the force of reaction expressed by $\frac{v^2}{\rho}$. The former of these, being a

tangential force, changes only the velocity; and the latter, being a normal force, changes only the direction, *i. e.* the former is merely an accelerating or retarding force; the latter merely a deflecting force.

That the deflecting force makes no change in the velocity, may perhaps require demonstration, more especially as it has been shown, that when the path is a polygon represented by $mm'm''$, &c. (Fig. 143.) the body having described the line mm', with the velocity v, enters on the line $m'm''$, with the velocity $v.\cos.\omega$. Whereby it would appear that at the angle m', a portion of the velocity is lost by deflection, whose value is $v(1-\cos.\omega)$; or, $v.\text{ver.sin.}\,\omega$. And if the figure described were actually a polygon, whether right lined or curved lined, the angles being then finite, those losses would have a finite value, and should, therefore, be estimated. But the case is different, when for the polygon we substitute its limiting curve. For the velocity lost in the direction of the tangent, is to that destroyed in the direction of the normal, as ver. sin. ω to sin. ω, *i. e.* as tan. $\frac{\omega}{2}$ to unity. Now the path being curvilinear, the angle ω is infinitely smaller than any right lined angle; and, therefore, tan. $\frac{\omega}{2}$ infinitely less than unity. The velocity destroyed at each instant in the direction of the normal, has been represented as the effect of a force, $\frac{v^2}{\rho}$, which must act during some definite time to produce any definite effect; and the velocity supposed to be destroyed in the direction of the tangent is infinitely less: wherefore, if this latter is ascribed to the action of any force, it must be to a force, such as cannot produce any effect in any definite time. Whence it follows, that there is no change made in the velocity, except that which is due to the tangential force; and that a body moving in a curve in consequence of an impulse, and not affected by

friction or the resistance of the medium, must for ever continue to move with the same unvaried velocity.

6. The perpendicular action of the constraining line or surface having no effect on the velocity, it might seem that in developing the theory of constrained motion, it would not be requisite to consider any force but that in the tangent; and undoubtedly this abbreviated process may be followed, when the body moves in a given curve without friction. But this is only a limited case of the general theory of constrained motion: for the tangential force depends on the path; and when the body is constrained by a surface, the path itself is to be investigated. In such cases, therefore, we cannot proceed from the tangential force to any of the objects of inquiry, viz. the time, the velocity, the path, or the pressure on the constraining surface. Moreover, in many cases, the tangential force depends not only on the form of the curve described, but also on the pressure; and, therefore, on the normal force, as when friction is to be considered. For these reasons, in developing the general theory of constrained motion, it seems proper to follow the usual method, which is also the most elegant, by including all the forces actually exerted, and then regarding the body as moving in free space. In this method there are three fundamental equations of the movement, by which the normal force of reaction will naturally be eliminated, whenever the quantity sought is independent of this force; whilst its introduction into those equations affords the readiest means of expressing its value, whenever this becomes the subject of investigation.

Because of some peculiarities of management, as the motions are restrained by lines or surfaces, it seems expedient to separate those parts of the general subject.

SECTION VI.

OF MOTION CONSTRAINED BY A GIVEN CURVE.

1. LET each of the forces, p, p', p'', &c. be resolved according to three rectangular axes; and let X, Y, and Z, denote, as usual, the sum of the resolved forces acting in directions parallel to the axes of x, y, and z, respectively. Also, let N denote the force of reaction in the normal, and θ, θ', θ'', the angles made by this normal with the same axes. We shall then have for the fundamental equations of the movement,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} + \mathrm{N}.\cos.\theta. \\ \frac{d^2y}{dt^2} &= \mathrm{Y} + \mathrm{N}.\cos.\theta'. \\ \frac{d^2z}{dt^2} &= \mathrm{Z} + \mathrm{N}.\cos.\theta''. \end{aligned}\right\} \quad (1)$$

Two of these equations will suffice for a plane curve: for then R, and consequently N, is in that plane. Wherefore, by taking two axes in the same plane, each of the terms of the third equation shall be cipher.

Further:

$$\cos.^2\theta + \cos.^2\theta' + \cos.^2\theta'' = 1. \quad (2)$$

Moreover, as $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, are the cosines of the angles made by the element of the curve, with the axes of x, y, and z, respectively; and as θ, θ', θ'', are the angles made by the normal with the same axes, it follows, that

$$\frac{dx}{ds}.\cos.\theta + \frac{dy}{ds}.\cos.\theta' + \frac{dz}{ds}.\cos.\theta'' = 0. \quad (3)$$

the first member being the expression for the cosine of the angle contained between the arc and normal, *i. e.* of a right angle.

By means of equations (1) and (3) the quantities N, $\cos.\theta, \cos.\theta', \cos.\theta''$, may be at once eliminated; and the result shall be a differential equation of the second order among the remaining quantities x, y, z, t, and the forces X, Y, Z; whereby the time may be expressed as a function of those forces and of the coordinates.

The equation thus obtained will also give the velocity. For, by substituting for dt its value $\frac{ds}{v}$, or $\frac{\sqrt{dx^2+dy^2+dz^2}}{v}$, the equation will be between v and the same quantities.

Equation (2) points to the method of eliminating the cosines by themselves, *i. e.* of reducing the co-efficient of N to unity; whereby the value of this force is at once expressed as a function of the remaining variables.

The coordinates in the expressions thus obtained may be reduced to one, by using the equation or equations of the curve for the elimination of the rest. Then, if X, Y, and Z are given functions of the coordinates, those forces being previously so expressed, the quantity sought, whether it be v, t, or N, shall be exhibited as a function of the remaining ordinate.

2. To proceed according to the method now described, let the first of the equations (1) be multiplied by dx, the second by dy, and the third by dz, and let these equations then be added; the co-efficient of N becomes cipher, as appears from equation (3), and the result is

$$\frac{dx.d^2x+dy.d^2y+dz.d^2z}{dt^2}=\text{X}.dx+\text{Y}.dy+\text{Z}.dz. \qquad (4)$$

If we differentiate the equation

$$\frac{dx^2+dy^2+dz^2}{dt^2}=\frac{ds^2}{dt^2},$$

we find that the first number of equation (4) is $ds.\frac{d^2s}{dt^2}$; and the equation becomes

$$\frac{d^2s}{dt^2}=\text{X}.\frac{dx}{ds}+\text{Y}.\frac{dy}{ds}+\text{Z}.\frac{dz}{ds}. \tag{5}$$

The terms in the second member of this last equation are the accelerating forces reduced to the tangent. The equation, therefore, shows that the tangential force is equal to the second differential coefficient of the space described, taken with respect to the time, as has been already proved with respect to unconstrained motion.

Equation (4), by integration, gives

$$\frac{dx^2+dy^2+dz^2}{dt^2}=2\int(\text{X}.dx+\text{Y}.dy+\text{Z}.dz)+\text{C}. \tag{6}$$

Using the equation or equations of the curve for elimination, the integral of the last equation gives the value of t, as a function of one of the coordinates. And here it may be observed, that as the expression for dt, to be obtained from equation (6), contains $\sqrt{dx^2+dy^2+dz^2}$, or ds, the time is dependent on the equations of the curve, *i. e.* the time taken to move from one point of the path to another, depends on the form of the curve described between those points; and is accordingly different for different curves described between two given points.

3. The first member of equation (6) being $\frac{ds^2}{dt^2}$, or v^2, that equation gives for the velocity,

$$v^2=2\int(\text{X}.dx+\text{Y}.dy+\text{Z}.dz)+\text{C}. \tag{7}$$

When $\text{X}dx+\text{Y}dy+\text{Z}dz$ is an exact differential of a function, $\phi(x, y, z,)$ of the three coordinates, the preceding equation assumes the form

$$v^2=2\phi(x,y,z,)+\text{C}.$$

And if v' denotes the velocity at any point whose coordinates are x', y', z', we shall have $v'^2=2\phi(x',y',z')+\text{C}$. Wherefore,

$$v^2-v'^2=2\phi(x,y,z)-2\phi(x',y',z').$$

This expression for the velocity, which is the same as that found for unconstrained motion in Sect. 3. Art. 5, is independent of the form of the curve. Wherefore, if different curves are described between the points whose coordinates are x', y', z', and x'', y'', z'', the body proceeding from one of those points with a certain velocity, would arrive at the other with a velocity which is the same for all those different courses. And if the curve is closed, the body shall ever return to the same point with the same velocity.

4. If $\text{R}=0$. *i. e.* if $\text{X}=0$. $\text{Y}=0$. $\text{Z}=0$. then

$$\int(\text{X}.dx+\text{Y}.dy+\text{Z}.dz)=0. \qquad \text{and } v^2=\text{C}.$$

Wherefore, when the body moves in conssquence of an original impulse, and is not solicited by any force, but that of the reaction of the constraining curve, the velocity is unvaried, or the same at every point in the line of its motion. This is agreeable to Art. 5. of last Sect. where it was shewn that there is no change in the velocity, except that made by the action of the tangential force.

5. If the body moving in the curve is urged by the force of gravity only, taking the axis of z vertical, and measuring the positive values of the parallel ordinates upwards, we shall have

$$\text{X}=0.\ \text{Y}=0.\ \text{Z}=-g. \text{ and } v^2=-2\int gdz+\text{C}=-2gz+\text{C}.$$

Let z' be the height at which the velocity is cipher, and we shall have $\text{C}=2gz'$. Wherefore,

$$v^2=2g(z'-z).$$

The value of the velocity, in this case, depends exclusively on the space $z'-z$; *i. e.* on the vertical distance from

the point where the velocity is cipher. Whence it follows, that if several bodies moving in different curves under the influence of the force of gravity, have equal velocities at any common height above a horizontal plane, their velocities shall be equal at any other given height above the same plane. The time, indeed, in this, as in all other cases, depends on the equation of the curve : but even in this respect, the case now under consideration is distinguished by this peculiarity, that the time depends on the relation between the curve and the vertical ordinates only ; as appears by substituting for v, its value in the equation $dt=\frac{ds}{v}$ which then becomes $dt=\frac{ds}{\sqrt{2g(z'-z)}}$. Wherefore, if there are two curves in which the relation between s and z, is the same, *i. e.* if for equal arches, the vertical ordinates are always equal, not only the velocities acquired, but also the times of the descent through those equal arches shall be equal; the initial velocity being the same in both : and this is true, however the curves may differ as to their horizontal coordinates. Two plane curves agreeing in the relation between the arc and one of the coordinates must be the same. Wherefore, the curves here spoken of can differ only when all but one are of double curvature. This diversity may be obtained by wrapping a plane curve round a vertical cylinder, whose base is a continued curve of any figure; whereby an indefinite variety of curves of double curvature may be produced from the same plane; and conversely, any one of the curves so produced may be opened into a plane: and from the foregoing equations it appears, that no change shall be made thereby in the velocity acquired in the descent through any arc, or in the time taken to describe it.

6. To find the pressure exerted at any point of the curve, let us suppose in the first place that there is no ac-

celerating force acting on the body. Equation (1), in this case becomes,

$$\frac{d^2x}{dt^2}=\text{N}\cos.\theta, \qquad \frac{d^2y}{dt^2}=\text{N}\cos.\theta', \qquad \frac{d^2z}{dt^2}=\text{N}\cos.\theta''.$$

Again, ds denoting the element of the curve described in the time dt, we have $ds=vdt$; in which ds is constant, since v is so, (Art. 4). Substituting then for dt its value $\frac{ds}{\text{v}}$, in the preceding equations, squaring and adding, we obtain

$$\text{N}=\frac{\text{v}^2\sqrt{(d^2x)^2+(d^2y)^2+(d^2z)^2}}{ds^2}.$$

But, ds being constant,

$$\rho=\frac{ds^2}{\sqrt{(d^2x)^2+(d^2y)^2+(d^2z)^2}}.$$

And we find, as before, (Sect. 5. Art. 2.), that the pressure arising from the inertia of the moving point is expressed by the formula

$$\text{N}=\frac{\text{v}^2}{\rho}.$$

To find the total pressure we must resolve the resultant, R, of the accelerating forces into two, one in the direction of the tangent, and the other normal to the curve. The former of these is the force which alters the velocity of the moving point, and the latter produces a pressure on the curve which is independent of the motion of the body. This may be called the *statical pressure.* It is to be combined with the centrifugal force, or the pressure arising from the inertia of the body, and the resultant will be the actual pressure sustained.

7. We may apply these results to the case of a heavy body, compelled to describe a vertical circle by the tension of a rod or wire, whose extremity is attached to a fixed point. Let C be the fixed point, (Fig. 146.), CA the vertical

passing through it, and CM any other position of the connecting rod. Let the force of gravity be represented by the vertical line MN, and let this be resolved into two, MQ in the direction of the tangent at the point M, and NQ perpendicular to it. The former of these is the force employed in accelerating the body; the latter is that portion of gravity which produces the strain on the fixed point. Letting fall the perpendicular Mn on the vertical radius, from the similarity of the triangles MNQ and CMn we have

$$\frac{\mathrm{MQ}}{\mathrm{MN}}=\frac{\mathrm{M}n}{\mathrm{MC}}, \qquad \frac{\mathrm{NQ}}{\mathrm{MN}}=\frac{\mathrm{C}n}{\mathrm{MC}}.$$

Wherefore, if g denote the force of gravity, f the accelerating force, and p the pressure on the axle; and if Mn and An the ordinate and abscissa of the arc terminating in the lowest point; be denoted by y and z, and the radius of the circle by r, we have

$$f=g\frac{y}{r}, \qquad p=g\frac{r-z}{r}.$$

The accelerating force, then, varies as the sine of the arc measured from the lowest point, and the pressure as its cosine.

The pressure just spoken of, however, is only the statical pressure, which would subsist if the force f were counteracted, and the body at rest. The whole pressure is obtained by adding to this the centrifugal force. This force, we have seen, is equal to $\frac{v^2}{r}$, or to $2g\,\frac{z'-z}{r}$; so that if m denote the mass of the body, the total pressure will be

$$mg\,\frac{r+2z'-3z}{r}.$$

If the body is let fall the rod being in the horizontal position, $z'=r$, and the tension at any point is $3mg\frac{r-z}{r}$. This,

in every part of the curve, is thrice the tension of the rod produced by the weight when supported by a force equal and opposite to the tangential force. If the body is supposed to descend from the vertical position without impulse; *i. e.* if it descends from B, the highest point of the circle, $z'=2r$: and the strain is $Mg\,\frac{(5r-3z)}{r}$. At the lowest point this is $5Mg$, or five times the weight.

If it is required to find the point at which the strain becomes equal to the weight, we have for this, $Mg\,\frac{(r+2z'-3z)}{r} = Mg$; which gives $z=\frac{2}{3}z'$. Accordingly, from whatever height the body descends without impulse, the strain becomes equal to the weight when it has descended through one-third of that height.

If it is required to find the point at which the pressure vanishes, its value is to be made equal to cipher. This gives $z=\frac{r+2z'}{3}$, for the height to which the body should rise, in order that the pressure may vanish. But it is possible that the body may never rise to this height; for z cannot surpass either z' or $2r$. This points to the greater and lesser limits of the magnitude to be assigned to z', in order that the body may rise to the height required by the formula. For making z successively equal to z' and $2r$, we have $z'=r$, $z'=\frac{5}{2}r$. If z' is less than r, the force of gravity resolved in the direction of the length of the pendulum is directed from the centre; and, therefore, conspires with the centrifugal force throughout the whole curve. And if z' is greater than $\frac{5}{2}r$, the centrifugal force is, in all parts of the curve, greater than the weight; as will appear by putting $z'=\frac{5}{2}r+\delta$, in the expression for that force. Being therefore greater than the pressure made by the weight on the centre, the connecting rod, throughout, suffers a tension; which is

the sum of those forces in the lower half, and their difference in the upper half of the circle. If z' is not less than r, or greater than $\frac{5}{2}r$, the body shall always attain to the height specified in the formula for a pressure equal to cipher: *i. e.* it shall rise to a point where the weight, resolved according to the length of the rod, becomes equal and opposite to the centrifugal force.

When the body rises above this point, the pressure due to the weight is greater than the centrifugal force: and being, moreover, directed towards the centre, the rod is compressed by the difference of those forces. Wherefore, if z' falls between the prescribed limits, and the heavy body is suspended by a flexible cord, it shall part from the circle at a point, whose vertical height above A is that given by the equation $z = \frac{r+2z'}{3}$. Being then unconfined by the cord, it shall move in a parabola, having a common tangent with the circle at the point of departure. The angle of elevation, as well as the velocity at this point, being given, the parabola is given.

SECTION VII.

OF THE SIMPLE PENDULUM.

1. The use of the pendulum as a regulator of those motions by which time is measured, and as affording the readiest and most correct measures of terrestrial gravitation, entitles this instrument to particular consideration. It consists of a heavy body, suspended by a rod from a horizontal axle, round which it is at liberty to move in the arch of a circle. In unfolding the theory of this instrument, it is convenient to regard the rod as being without weight, and the body attached to it as a material point. The instrument, so considered, is called by writers on mechanics a simple pendulum. In reality, the rod, like all other material substances, is heavy, and the body suspended by it of some certain bulk; but as the time of the vibration of a pendulum depends not merely on its length, measured from the axle to the remote extremity of the appended body, but also on the form of the whole pendulum; or more properly, on the distribution of the matter of which it consists, it is found expedient to reduce every real pendulum to the simple pendulum: *i. e.* to estimate its length by that of the simple pendulum which would perform its oscillations in the same time. The mode of making this reduction shall be shown in a subsequent part of this treatise. For the present we shall consider the case of the simple pendulum.

2. It has been already proved, that when a body is constrained to move in a curve whose plane is vertical, by the influence of gravity alone, the velocity acquired at any point is that due to the height from which it has descended, mea-

sured in the vertical. The elementary proof of this is exceedingly simple. The curve being regarded as a polygon with an infinite number of sides, the body may be supposed to descend down a system of planes inclined to the horizon at different angles; and the planes being infinite in number, no part of the velocity acquired shall be lost in passing from one to the other. Wherefore, l, l', l'', &c. denoting the lengths, and h, h', h'', &c. the heights of the planes, beginning with the uppermost, and v, v', v'', &c. the velocities at the lower extremities of those planes, there will be for the fall down the plane l, $v^2 = 2gh$. And v being the velocity with which the body enters on the second plane, there is at the end of this plane, $v'^2 - v^2 = 2gh'$. And in like manner, at the end of the third plane, $v''^2 - v'^2 = 2gh''$. Wherefore, by addition, $v''^2 = 2g(h + h' + h'')$. The same proof extends to any number of planes: accordingly, if z' denotes the vertical height of the point from which the body is suffered to descend; and z that of the point to which it falls; both measured upwards from the same horizontal plane. The velocity will be expressed by the formula

$$v = \sqrt{2g(z' - z)}.$$

Now the condition of a body suspended by a rod is the same as if its movement were constrained by a vertical circle: the tension of the rod taking place of the pressure, and the reaction of the fixed point that of the constraining curve. Wherefore, let BMAM'B be the circle whose plane is vertical, (Fig. 147.); A the lowest point; and AB the vertical diameter. If the body is suffered to descend from M, its velocity at any point, N, is had by drawing the horizontal lines MM' and NN', meeting the vertical diameter at E and F; for the velocity at N, is that due to the height EF. The body having descended to the lowest point A, where it acquires a velocity due to the height EA, begins to ascend against the force of gravity, in the other branch of the curve AM'; and as it appears by the equation, that in the two branches the velocities are equal at

equal heights, it follows, that the velocity in the ascending branch shall not be destroyed until the body arrives at M′. The velocity at this point being cipher, the body shall descend by its weight to A; where regaining the velocity with which, in its progress, it had passed through this point, it shall ascend to M: and thus the body shall go and return for ever through the same arc MAM′, supposing no retardation from friction, or from the resistance of the medium through which it moves.

3. The time, dt, of describing any element of the arc, ds, is given by the formula

$$dt = \frac{-ds}{v};$$

in which dt and ds are affected with opposite signs, because as the time increases, the arc measured from the lowest point diminishes. Substituting in this expression for v its value found in the preceding article, we have

$$dt = \frac{-ds}{\sqrt{2g(z'-z)}}.$$

When the arc of vibration is indefinitely small, this formula is readily integrated. Let s and s' be the arcs measured from the lowest point, corresponding to the ordinates z and z'; these arcs, being indefinitely small, are equal to their chords, and we have

$$s^2 = 2rz, \qquad s'^2 = 2rz'.$$

Substituting then for $2z$, $2z'$, their values $\frac{s^2}{r}$, and $\frac{s'^2}{r}$, the preceding formula becomes

$$dt = -\sqrt{\frac{r}{g}}\frac{ds}{\sqrt{s'^2 - s^2}}$$

The integral of which is

$$t = \sqrt{\frac{r}{g}} \text{ arc } \left(\cos. = \frac{s}{s'}\right).$$

which needs no correction, as $t = o$, when $s = s'$.

The time of descent to the lowest point is had by making $s=0$; and as arc $(\cos.=0)=\frac{\pi}{2}$, this time is

$$\frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

The time of an oscillation, or of the whole descent and subsequent ascent, is double of this, and if its value be denoted by T,

$$\text{T}=\pi\sqrt{\frac{r}{g}}.$$

This expression being independent of the magnitude of the arc described, it follows that the time of oscillation will be the same, however the arc of vibration be varied, provided that that arc is indefinitely small. It is this property which renders the pendulum an instrument of such value in the measurement of time.

Gravity being constant at any given place, it follows from the same formula, that the times of vibration of different pendulums vary as the square roots of their lengths.

4. Many curious relations are at once deduced from the expressions just obtained. Thus, the time of the fall in the vertical through a space equal to $\frac{1}{2}r$ being $\sqrt{\frac{r}{g}}$, it follows, that the time of vibration is to that of the fall through half the length of the pendulum, as $\pi : 1$; *i. e.* as the circumference of a circle to its diameter.

The time of the fall down the vertical diameter is $2\sqrt{\frac{r}{g}}$. And this being the same as the time of descent through any chord terminating at the lowest point, it follows, that the time of describing a very small circular arc ending at the lowest point, is to that of the descent down its chord,

as $\frac{\pi}{2}\sqrt{\frac{r}{g}}$ to $2\sqrt{\frac{r}{g}}$: *i. e.* as π to 4, or as the circumference of a circle to four times its diameter.

From the last equation it also appears, that if two pendulums of different lengths are set to vibrate in small circular arcs, the times shall be as the square roots of the lengths; or the lengths as the squares of the times: *i. e.* inversely as the squares of the numbers of vibrations performed by the two pendulums in the same time. Wherefore, n denoting the number of vibrations made in a given time, $n^2.r$ shall be a constant quantity: *i. e.* the same for all pendulums vibrating in small circular arches, at the same part of the surface of the earth. Accordingly, given the length of the pendulum, and the number of vibrations made by it in a given time, the length of a second's pendulum is thereby known. For let r' be the length of the given pendulum, and r that of the second's pendulum; also, let n' be the number of the vibrations made by the former in n seconds, we have $n^2.r = n'^2.r'$; whence, $r = \frac{n'^2.r'}{n^2}$.

Hence, also, knowing the daily or weekly loss or gain of a pendulum clock, the correction to be made in the length of the pendulum is immediately ascertained. For let n be the number of seconds in a day or in a week, and $n \pm dn$ the number shown by the clock; there is $n^2r = (n \pm dn)^2.r' = n^2r' \pm 2n.dn.r'$. omitting the third term of the square of $n \pm dn$, as being inconsiderable with respect to the second. This equation gives $n^2(r-r') = \pm 2n.dn.r'$, or $r - r' = \pm\frac{2dn}{n}r'$; *i. e.* the correction of the length of the pendulum is had by multiplying that length into a fraction, whose numerator is twice the error in time, and whose denominator is the time in which it had accumulated.

The equation shews that $r-r'$ is positive or negative,

as the clock gains or loses; and that the pendulum is to be lengthened or shortened accordingly.

5. Besides the advantages derived from the pendulum for the measurement of time, we obtain from it the readiest, as well as the most precise method of satisfying our inquiries relative to some of the most important subjects in Physical Science.

Thus, the equation $\text{T}=\pi\sqrt{\frac{r}{g}}$, exhibiting the dependence of any one of the quantities T, r, and g on the remaining two, suggests the mode of finding whether the accelerating force of gravity acts alike on all terrestrial substances, or with an intensity which is different for the different species of substances. Now in the same part of the earth, pendulums of equal lengths are found to be perfectly isochronous, however different the kinds of matter of which they consist. This fact is ascertained with the greatest precision: for whilst two pendulums are suffered to descend together, if there were the smallest difference in the times of their vibrations, this difference, after a number of vibrations, would be multiplied by that number, and would thereby become plainly perceptible; the pendulums not moving in the corresponding points of their paths, *i. e.* the rods not remaining parallel. The experiment, therefore, shows that when the lengths are equal, the times are precisely equal; and the result being the same, however the substances be varied, it follows, that all are equally accelerated by gravity; or that this force acts on all bodies, in proportion to their quantities of matter.

The equation $\text{T}=\pi\sqrt{\frac{r}{g}}$ gives $g=\frac{\pi^2.r}{\text{T}^2}$. If the pendulum vibrates seconds, T is unit; and l denoting the length of such a pendulum, the equation is $g=\pi^2 l$, which expresses the velocity generated by gravity in one second, by the space which would be described with that velocity continued

uniformly for that time. The space through which a body falls from rest in one second is one-half of this, or $\pi^2 \frac{l}{2}$; which shows that the space described in one second, by a body falling freely from a state of rest, is to half the length of the second's pendulum, in the duplicate ratio of the periphery of a circle to its diameter.

The foregoing expressions require only the substitution of their numerical values for π and l. Now in these latitudes, l is found to be 39.14 inches. Wherefore, substituting this number for l, and for π its value, viz. 3.14159, the value of g will be 32.19 feet: and the space described by a body falling freely from rest, will be 16.09 feet, or nearly 16 feet and one inch.

But the force of gravity, though constant in the same place, is not of the same intensity at all parts of the earth's surface. In Sect. 5. Art. 3. it has been shewn, that the effective force of gravity is diminished from the poles to the equator, by the centrifugal force. Hence it follows, that if the earth were a fluid mass, the columns extending from the centre to the surface would vary in length; those nearer to the plane of the equator rising, to balance by their greater mass, those in which the effective gravity is greater. There is then a further change of the effective gravity arising from figure. It is shewn by the writers on Physical Astronomy, that the figure, which the earth would thus assume, would be an oblate spheroid; and that in proceeding from the equator towards the pole, the increment of gravity on the surface of a homogeneous body of this figure would vary as the square of the sine of latitude: *i. e.* according to the same law as the increment immediately due to the centrifugal force. Wherefore, G denoting the effective gravity at the equator, and mG the whole of the excess of that at the poles, the increment at any latitude, λ, would be $mG \sin.^2\lambda$; and the effective gravity at that latitude would be

$$g = G(1 + m \sin.^2\lambda).$$

From this equation the value of m is known when the ratio of g to G is found: and this is the ratio of the lengths of two isochronous pendulums, of which, one vibrates at the latitude λ, and the other at the equator. Wherefore, l and L being those lengths, the equation is

$$l = L(1 + m \sin.^2\lambda).$$

As it may not be convenient to find the length of the pendulum at the equator, L is to be exterminated; for which purpose, a second equation will be requisite. Wherefore, putting l' for the length of a second's pendulum at any other latitude λ', there is, in like manner, $l' = L(1 + m \sin.^2\lambda')$; and by division $\frac{l'}{l} = \frac{1 + m \sin.^2\lambda'}{1 + m \sin.^2\lambda}$. Which gives

$$m = \frac{l' - l}{l \sin.^2\lambda' - l' \sin.^2\lambda}.$$

The number of vibrations performed in a given time can be observed with more ease and precision than the length of the pendulum: and the lengths of isochronous pendulums being as the squares of the numbers of vibrations made by the same pendulum in a given time, at those different latitudes, the formula may be written thus:

$$m = \frac{n'^2 - n^2}{n^2 \sin.^2\lambda' - n'^2 \sin.^2\lambda}.$$

Wherein, n and n' denote the numbers of the vibrations made in a given time by the same pendulum, at the latitudes λ and λ'.

By making $n' = n + dn$, and neglecting the terms which are inconsiderable with respect to the rest, the formula becomes

$$m = \frac{2dn}{n(\sin.^2\lambda' - \sin.^2\lambda)} = \frac{2dn}{n \sin.(\lambda' + \lambda) \sin.(\lambda' - \lambda)}.$$

Experiments conducted on these principles give $m = \frac{1}{191}$.

Shewing that the difference of gravity at the equator and the poles is the 191st part of the equatorial gravity.

The pendulum by giving the value of m, gives also the ellipticity of the earth, *i. e.* the difference of the equatorial and polar semidiameters divided by the former. It is proved, indeed, by the writers before referred to, that if the earth were of uniform density, the ellipticity would be $\frac{5}{4}$ of the fraction expressing the ratio of the centrifugal force to gravity at the equator: and that the same would also be the value of m; *i. e.* q denoting the centrifugal force at the equator divided by gravity, and e the ellipticity, there would be $e=\frac{5}{4}q=m$. Now, $q=\frac{1}{289}$, wherefore, $\frac{5}{4}q=\frac{1}{231}$, for the value of e or m. This is Newton's discovery.

The supposition of a uniform density is in the highest degree improbable; and as the value of m depends on the constitution of the earth, with respect to the density of its interior parts, the value of m, found on that supposition, is not to be regarded even as an approximation to the truth. But Clairaut has shewn, that according to whatever law the density may vary from the surface to the centre, the above value of e and m, found on the supposition of a uniform density, is an arithmetical mean between the real values of those quantities: *i. e.* that

$$e+m=\tfrac{5}{2}q.$$

substituting in this equation for m and q their values, we have

$$e=\tfrac{1}{289}.$$

The value of e can, indeed, be discovered by several other methods, but not with the same facility, or certainty of its accuracy. The most obvious of those methods is that by the measurement of the meridional arcs; and the result obtained by the most accurate of these measurements is $\frac{1}{312}$. The ellipticity deduced by Laplace from the observed magnitude of two of the lunar inequalities is $\frac{1}{305}$.

Clairaut has shown, that for a density increasing towards the centre, the value of e would be less, and consequently,

that of m greater than $\frac{5}{4}q$, or $\frac{1}{231}$; that for a uniform density there would be $e=\frac{5}{4}q$; and for a density infinite at the centre, $e=\frac{q}{2}$. Accordingly, for the same limits, there would be $m=\frac{5}{4}q$ and $m=2q$. The values found by observation are between those limits, and, therefore, show that the density of the earth increases towards the centre: a conclusion antecedently probable, and further confirmed by experiments made on the attractions of mountains.

6. Hitherto the arch of descent has been regarded as indefinitely small. To deduce its value generally we must substitute for ds its value $\frac{rdz}{\sqrt{2rz-z^2}}$, in the equation

$$dt=\frac{-ds}{\sqrt{2g(z'-z)}}.$$

and we find

$$dt=\frac{-r.dz}{\sqrt{2g(z'-z)(2rz-z^2)}}.$$

This equation cannot be integrated in finite terms. To expand it into a series, let the quantity under the radical sign be multiplied, and divided by $2rz$. This changes the equation to

$$dt=\tfrac{1}{2}\sqrt{\frac{r}{g}}.\frac{-dz}{\sqrt{z'z-z^2}}\times\frac{1}{\sqrt{1-\frac{z}{2r}}}.$$

Developing the factor $\left(1-\frac{z}{2r}\right)^{-\frac{1}{2}}$ into the series

$1+\frac{1}{2}.\frac{z}{2r}+\frac{1}{2}.\frac{3}{4}.\frac{z^2}{4r^2}+\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{z^3}{8r^3}+$ &c. we have

$$dt=\tfrac{1}{2}\sqrt{\frac{r}{g}}.\frac{-dz}{\sqrt{z'z-z^2}}\left\{1+\tfrac{1}{2}\left(\frac{z}{2r}\right)+\tfrac{1}{2}.\tfrac{3}{4}\left(\frac{z}{2r}\right)^2+\right.$$
$$\left.\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}\left(\frac{z}{2r}\right)^3+\text{\&c.}\right\}$$

The terms to be integrated are of the form $\int\frac{-z^m.dz}{\sqrt{z'z-z^2}}$,

the exponents m being the natural numbers beginning with cipher. Wherefore, putting the several integrals, $\int \frac{-z^m.dz}{\sqrt{z'z-z^2}} = A_m$, $\int \frac{z^{m-1}.dz}{\sqrt{z'z-z^2}} = A_{m-1}$, &c. - - - - - - - - - - $\int \frac{-dz}{\sqrt{z'z-z^2}} = A_0$, the time of the entire vibration will be

$$T = \sqrt{\frac{r}{g}}.\left\{ A_0 + \tfrac{1}{2}\left(\frac{1}{2r}\right) A_1 + \tfrac{1}{2}.\tfrac{3}{4}\left(\frac{1}{2r}\right)^2 A_2 + \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}\left(\frac{1}{2r}\right)^3 A_3 + \text{ \&c.} \right\} \quad (1)$$

Each of the terms of this series after the first, is derived from the preceding term: for,

$$\int \frac{-z^m.dz}{\sqrt{z'z-z^2}} = \frac{\sqrt{z'z-z^2}}{m}.\,z^{m-1} + \frac{z'(2m-1)}{2m}\int \frac{-z^{m-1}.dz}{\sqrt{z'z-z^2}}.*$$

These integrals are to be taken between $z = z'$, and $z = 0$; and at each of these limits, the first term of the second member vanishes. Wherefore,

$$\int \frac{-z^m.\,dz}{\sqrt{z'z-z^2}} = z'\,\frac{(2m-1)}{2m}\int \frac{-z^{m-1}.\,dz}{\sqrt{z'z-z^2}},$$

This formula exhibits the dependence of each of the integrals on the one preceding. Wherefore, all are known when the first, viz. A_0, or $\int \frac{-dz}{\sqrt{z'z-z^2}}$, is known. But $\int \frac{-dz}{\sqrt{z'z-z^2}} = \text{arc}\left(\cos. = \frac{2z-z'}{z'}\right)$. which, taken from $z = z'$, to $z = 0$, is $\int \frac{-dz}{\sqrt{z'z-z^2}} = \pi$. Wherefore, making m successively, 1, 2, 3, 4, &c., we have

$$A_1 = \tfrac{1}{2}z'.\pi.$$
$$A_2 = \tfrac{3}{4}z'\,A_1 = \tfrac{1}{2}.\tfrac{3}{4}.z'^2.\pi.$$
$$A_3 = \tfrac{5}{6}z'\,A_2 = \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.z'^3.\pi.$$
&c. &c. &c.

* See Lardner's Integral Calculus, Sect. 4. Art. 222.

Making these substitutions in equation (1), it becomes

$$\text{T} = \pi\sqrt{\frac{r}{g}}\left\{1 + (\tfrac{1}{2})^2\left(\frac{z'}{2r}\right) + (\tfrac{1}{2}.\tfrac{3}{4})^2\left(\frac{z'}{2r}\right)^2 + (\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6})^2\left(\frac{z'}{2r}\right)^3 + \ \&c.\right\}.$$

This series, proceeding according to the powers of $\frac{z'}{2r}$, always converges, inasmuch as z' is less than $2r$: and it converges the more rapidly, the smaller the quantity $\frac{z'}{2r}$. If the arc is very small, the first term will suffice: which gives $\text{T} = \pi\sqrt{\frac{r}{g}}$, the same conclusion to which we arrived by taking the arcs for their chords. Unless the arc of vibration is very considerable, the two first terms will suffice: of which, the second, called the correction, is $\frac{z'}{8r}$. Denoting the semi-angle by θ, $r\theta$ is the arch whose verse sine is z'; and this arch being confounded with its chord, there is $z' = \frac{r^2.\theta^2}{2r} = \frac{r.\theta^2}{2}$. *q.p.* and $\frac{z'}{8r} = \frac{\theta^2}{16}$, giving for the corrected value of the time of the vibration,

$$\text{T} = \pi\sqrt{\frac{r}{g}}\left(1 + \frac{\theta^2}{16}\right).$$

7. If the curve in which the body is constrained to move be the cycloid, having its axis vertical, the relation between s and z is expressed, generally, by the equation,

$$s^2 = 4az.$$

in which a is the diameter of the generating circle. Let s' denote the arc corresponding to the ordinate z', and let $\frac{s^2}{4a}$, $\frac{s'^2}{4a}$ be substituted for z and z', in the expression $\sqrt{2g(z' - z)}$; we then have

$$v = \sqrt{\frac{g}{2a}} \cdot \sqrt{s'^2 - s^2}.$$

The expression for the time is derived from the equation

$$dt = \frac{-ds}{v} = -\sqrt{\frac{2a}{g}} \cdot \frac{ds}{\sqrt{s'^2 - s^2}}.$$

which, by integration, gives

$$t = \sqrt{\frac{2a}{g}} \cdot \text{arc}\left(\cos. = \frac{s}{s'}\right).$$

This requires no correction; since for $t = 0$, there is $s = s'$, and therefore $\text{arc}\left(\cos. = \frac{s}{s'}\right) = 0$.

At the lowest point of the arc, s vanishes, and arc $(\cos. = 0) = \frac{\pi}{2}$. Wherefore the time of the descent to this point is expressed by the equation $t = \frac{\pi}{2}\sqrt{\frac{2a}{g}}$. And if T denote the time of the entire vibration

$$\text{T} = \pi \sqrt{\frac{2a}{g}}.$$

The time of vibration, therefore, is independent of the magnitude of the arc described; and the property of *tautochronism*, which belongs to the circle, only when the arcs of vibration are indefinitely small, holds universally for the cycloid.

The velocity and time may now be represented by a geometrical construction in the same way as in Sect. 2. Art. 7. Thus in (Fig. 148.) *ll* being the whole arc of vibration, let a tangent be drawn at V the lowest point, and let a circle, *l*′V*l*′. be described, whose centre is V, and radius V*l*′ equal to the cycloidal arc V*l*. Then, if V*m*′ is taken from the centre, equal to the arc V*m*, the ordinate erected at *m*′ or

$m'q = \sqrt{\mathrm{v}l'^2 - \mathrm{v}m'^2} = \sqrt{\mathrm{v}l^2 - \mathrm{v}m^2} = \sqrt{s'^2 - s^2}$. Wherefore, $v = \left(\sqrt{\frac{g}{2a}}\sqrt{s' - s^2}\right) = \sqrt{\frac{g}{2a}}.m'q$. Accordingly, the velocity at any point, m, varies as $m'q$, the ordinate of the circle raised at the corresponding point of its diameter. And $\mathrm{v}m'$ being the cosine of the arch $l'q$, to the radius $\mathrm{v}l'$, there is arc $\left(\cos. = \frac{s}{s'}\right) = \frac{\text{arc } l'q}{\mathrm{v}l'}$. Whence, $t = \sqrt{\frac{2a}{g}}$. arc $\left(\cos. = \frac{s}{s'}\right) = \sqrt{\frac{2a}{g}}.\frac{\text{arc } l'q}{\mathrm{v}l'}$. showing that the time of the movement from l, to any point, m, varies as the circular arch $l'q$, cut off by the ordinate $m'q$.

Owing to the property of tautochronism, which has been shown to belong to the cycloid, it has been supposed, that a pendulum made to oscillate in this curve, was peculiarly fitted for the regulation of the movements of clock work. Induced by this supposition, the mathematicians of the seventeenth century have bestowed more than ordinary attention on the properties of this curve: and though their labours on this subject have not been rewarded by the attainment of the proposed end, it must be admitted, that they have materially contributed to the advancement of mathematical science.

The method of causing a pendulous body to move in any given curve, is naturally suggested by the generation of a curve from its evolute. The evolute of a cycloid consists of two semi-cycloids, equal to the halves of the given cycloid, or involute, and having their vertices at the extreme points of its base. Thus, (Fig. 148.) the semi-cycloids OA, OB, equal to BV, AV, and similarly placed, shall constitute the evolute of the cycloid AVB. Accordingly, a body may be made to vibrate in a given cycloid, by placing two semi-cycloidal plates, each equal to half the given cycloid in the

positions shewn in the figure; and then suspending the body by a flexible string whose length is OV, or twice the axis of the cycloid. For as the string applies itself alternately to the plates OA, OB, the body shall always be found in the curve AVB.

But for the regulation of the movements of clock work, it is not requisite that the pendulum should vibrate in the arc of a cycloid: neither is the contrivance, whereby it is made to move in that curve, fitted to insure tautochronism. For the point at which the string parts from the cycloidal plate is the temporary centre of motion. Consequently, the part which at any instant hangs freely, together with the appended weight, is to be regarded as the vibrating body at that instant; rejecting that part of the string whose motion is prevented by the cycloidal plate. The part of the pendulum which hangs freely, being variable both in figure and weight, the point at which it may be supposed to be concentred, or, as it is named, the centre of oscillation is not necessarily an invariable point of the substance of the pendulum. Consequently, the contrivance, however well fitted to cause any given point of the body to move in a cycloid, is not fitted to give this motion to the ever varying centre of oscillation. For these and other reasons the circular arc is invariably preferred.

8. It is a problem of some interest to determine whether the property of *tautochronism*, which has been found to belong to the cycloid, is possessed exclusively by that curve.

The time of descent in any curve to the lowest point is found by integrating the equation $dt = \frac{ds}{\sqrt{2g(z'-z)}}$ between the limits $z=z'$, and $z=0$. Wherefore, if that time be denoted by t', there is

$$t' = \int_0^{z'} \frac{ds}{\sqrt{2g(z'-z)}}.$$

And in order that the curve may possess the property of tautochronism, the value of this integral must be independent of z', the ordinate of the initial arc. We are then to inquire what is the relation of s and z, which will fulfil this condition.

Let s be supposed to be developed according to the ascending powers of z, so that

$$s = az^{\alpha} + bz^{\beta} + cz^{\gamma} + \&c.$$

in which the coefficients and exponents are indeterminate. It is evident then, in the first place, that the exponents α, β, γ, &c. are all greater than cipher, since s and z vanish together. Differencing this expression, and substituting the value of ds in the formula given above

$$t' = a\alpha \int_{o}^{z'} \frac{z^{\alpha-1} dr}{\sqrt{2g(z'-z)}} + \&c.$$

or, making $z = z'u$,

$$t' = a\alpha z'^{\alpha-\frac{1}{2}} \int_{o}^{1} \frac{u^{\alpha-1} du}{\sqrt{2g(1-u)}} + \&c.$$

in which there will be as many terms as in the development of s. Now, in order that this expression should be independent of z', the coefficient of z' in one of these terms must be cipher, and all the other terms must vanish. We must have, therefore,

$$\alpha = \tfrac{1}{2}.$$

And since the exponents β, γ, &c. are all positive, as well as the definite integrals taken between the limits 0, 1, the other terms cannot vanish unless the coefficients b, c, &c. vanish; so that there is

$$b = 0, \quad c = 0, \quad \&c.$$

and the equation of the curve is reduced to

$$s = az^{\frac{1}{2}}.$$

which is the equation of the cycloid. The cycloid therefore is the only plane curve possessing the property of tautochronism.

9. On comparing the time of descent down an indefinitely small arc of a circle, terminating at the lowest point with that down its chord, it was found (Art. 4.) that the latter is to the former as the circumference of a circle to four times its diameter. From this example it appears, that the time of descent from one point to another, not in the same vertical, is not the shortest, for the shortest line that can be drawn through those points; and it becomes a problem to find the line of swiftest descent.

Let A and B be the two points, and AmB the curve of swiftest descent from the one to the other, (Fig. 149.) Then, if two other points are taken in this curve, as m and m', the intercepted arc mbm' also must be that of swiftest descent from one of those points to the other. For, let it be supposed that any other arc, as mcm', is described in a lesser time than the arc mbm': the velocity at m' being that due to the vertical height An', is the same for both; and, therefore, the body would take the same time to describe the remaining arc m'B, by whatever course it arrives at m'. Wherefore, the difference of the times of describing the whole arcs Ambm'B, and Amcm'B, shall be that of describing the parts mbm', mcm', which are not coincident. Accordingly, the time of descent through the arc Amcm'B, would be shorter than that of the descent through Ambm'B, contrary to the supposition. If, then, the latter arc is that of swiftest descent between its extreme points, the same property must belong to each of its parts as mbm'. As this inference is independent of the magnitude of that part, mm' may be supposed indefinitely small: and the body arriving at m, with a velocity due to the vertical height An, must describe the arc mbm' in a less time than any other arc between the limits m and m'.

To express this condition, let nn' be divided at the point e into two equal parts dz; then putting $An=z$, $mn=y$, $Am=s$, the velocity at m shall be $\sqrt{2gz}$, and the time taken to describe mb or ds shall be $\frac{ds}{\sqrt{2gz}}$. In like manner, putting $Ae=z+dz=z'$, $be=y'$, and $Ab=s+ds=s'$, the time of describing the arc bm' or ds', shall be $\frac{ds'}{\sqrt{2gz'}}$. And the time taken to describe the whole arc mm' shall be

$$\frac{ds}{\sqrt{2gz}}+\frac{ds'}{\sqrt{2gz'}}.$$

The condition requires that this should be a minimum: *i. e.* less than for any other curve that can be drawn between the points m and m', *i. e.*

$$\delta\left(\frac{ds}{\sqrt{2gz}}+\frac{ds'}{\sqrt{2gz'}}\right)=0.$$

The character δ denotes the variation of the position of a point from one curve to another; differing in this respect from the character d, which denotes the change of position in the same curve. In the present instance, z, z' are independent of those variations; being the same whether the curve is mbm' or mcm'. Therefore, $\delta z=0$. $\delta z'=0$. Accordingly, $\frac{\delta.ds}{\sqrt{z}}+\frac{\delta.ds'}{\sqrt{z'}}=0$. But $ds=(dz^2+dy^2)^{\frac{1}{2}}$, and because dz has no variation, $\delta.ds=\delta(dz^2+dy^2)^{\frac{1}{2}}=\frac{dy.\delta.dy}{\sqrt{dz^2+dy^2}}=\frac{dy.\delta.dy}{ds}$. And in like manner, $\delta.ds'=\frac{dy'.\delta.dy'}{ds'}$. Therefore,

$$\frac{dy.\delta.dy}{ds\sqrt{z}}+\frac{dy'.\delta dy'}{ds'\sqrt{z'}}=0. \qquad (2)$$

Now, whatever be the curve described between the points m, m', the difference between the ordinates at those

points is constant, *i. e.* $dy+dy'$ is a constant quantity: wherefore, $\delta(dy+dy')=0$. *i. e.* $\delta.dy=-\delta.dy'$, which changes equation (2) to

$$\frac{dy}{ds\sqrt{z}}-\frac{dy'}{ds'\sqrt{z'}}=0.$$

The second term of this equation being what the first becomes, when the point m is changed for the point b, the equation expresses the same thing as $d\left(\frac{dy}{ds\sqrt{z}}\right)=0$. Wherefore, $\frac{dy}{ds\sqrt{z}}=\text{c}$. But $\frac{dy}{ds}$ is the sine of the angle which the arc ds makes with the axis of z. And where the arc is horizontal, this angle is right; and, therefore, its sine $=1$. Wherefore, putting a for the unknown abscissa of the point where the tangent is horizontal, there will be $\text{c}=\frac{1}{\sqrt{a}}$. Whence

$$\frac{dy}{ds\sqrt{z}}=\frac{1}{\sqrt{a}}, \qquad \text{or } \frac{dy}{ds}=\sqrt{\frac{z}{a}}.$$

Eliminating dy between this and the equation $ds^2=dy^2+dz^2$, there is $ds=dz\sqrt{\frac{a}{a-z}}$. Whence, $s=-2\sqrt{a(a-z)}+\text{c}'$. For $s=0$, there is $z=0$. Wherefore, $\text{c}'=2a$, and

$$s=2a-2\sqrt{a(a-z)}.$$

This is the equation of a cycloid, the arc s being measured from the horizontal base, and a being the axis.

To find a, the axis of the cycloid, or the diameter of its generating circle, let a cycloid be described on the base AD, of any length, measured from the point A, on the horizontal line AY: and let this curve intersect the right line AB at K.

Then drawing KD, and from B the parallel BE, AE shall be the base equal to πa.

This construction is founded on the principle, that all cycloids are similar curves. Such are all curves defined by the same equation, provided that equation involves but one constant quantity.

SECTION VIII.

OF MOTION CONSTRAINED BY A SURFACE.

1. The case of motion restrained by a surface, is to be treated much in the same way as that of motion restrained by a line. The difference of conditions will, however, create some difference in the theory: for which reason, it seemed expedient to reserve this part of the subject for a distinct consideration.

The fundamental equations are those already given in Sect. 6. Art. 1. And the expressions for the time and velocity are derived from them by the method employed in that article. But if it is desired to express the relation between the time or the velocity, and one of the coordinates, there being but one equation of the surface, it will be necessary to use a second equation to be obtained by eliminating N between two of the equations (1). By this proceeding, the cosines are not made to reappear, inasmuch as they may be expressed by the co-efficients of dx, dy, dz, in the differential equation of the surface. That equation being represented generally by $ldx+mdy+ndz=0$. gives

$$\cos.\theta=\frac{l}{\sqrt{l^2+m^2+n^2}}. \qquad \cos.\theta'=\frac{m}{\sqrt{l^2+m^2+n^2}}.$$

$$\cos.\theta''=\frac{n}{\sqrt{l^2+m^2+n^2}}.\ *$$

Whereby the three equations, (1), become

* See Lardner's Differential and Integral Calculus.

$$\frac{d^2x}{dt^2}=\text{X}+\text{N}.\frac{l}{\sqrt{l^2+m^2+n^2}}.$$

$$\frac{d^2y}{dt^2}=\text{Y}+\text{N}.\frac{m}{\sqrt{l^2+m^2+n^2}}.$$

$$\frac{d^2z}{dt^2}=\text{Z}+\text{N}.\frac{n}{\sqrt{l^2+m^2+n^2}}.$$

Multiplying the first of these by dx, the second by dy, and the third by dz, we have, as before,

$$\frac{dx.d^2x+dy.d^2y+dz.d^2z}{dt^2}=\text{X}.dx+\text{Y}.dy+\text{Z}.dz. \qquad (a)$$

From which we get the expressions for the time and velocity, as in Sect. 6. Articles 3 and 4, with the other consequences there derived, relative to those quantities.

A second independent equation is obtained by eliminating N between any two of the fundamental equations. Thus, multiplying the first by m, and the second by l, and then subtracting one from the other, we have

$$\frac{m.d^2x-l.d^2y}{dt^2}=m\text{X}-l\text{Y}. \qquad (b)$$

By means of this and of the equation of the surface, two of the coordinates may be eliminated from equation (a), and t or v thereby expressed as a function of one of the coordinates only.

The trajectory itself is not given; but by eliminating dt between the equations (a) and (b), we get an equation, which, with that of the surface, exhibiting the relations of the coordinates, furnish the two equations of the trajectory.

2. The expression for the pressure is found by adding the three fundamental equations multiplied by l, m, n, respectively; and substituting for dt^2 its value, viz. $\frac{ds^2}{v^2}$. By this process we get

$$\text{N} = \frac{v^2}{ds^2} \cdot \frac{l.d^2x + m.d^2y + n.d^2z}{\sqrt{l^2+m^2+n^2}} - \frac{l\text{X} + m\text{Y} + n\text{Z}}{\sqrt{l^2+m^2+n^2}}.$$

This is the reaction; to which the pressure on the surface is equal and opposite.

Instead of applying to this formula, the pressure may be conveniently derived from the forces R and $\frac{v^2}{\rho}$. In the use of which method, it is to be observed, that when the motion is restrained by a surface, the reaction at any point of the trajectory is in the perpendicular to the surface: and as there is but one normal at each point of the surface, the forces R and $\frac{v^2}{\rho}$, are to be reduced to this direction. Wherefore, putting α, α' for the angles made by the normal with the direction of R, and with that of the radius of the circle of curvature at the same point, the pressure is $\text{R}.\cos.\alpha + \frac{v^2}{\rho}.\cos.\alpha'$.

3. Some important conclusions are to be derived by considering the other components of the same forces. Wherefore, let the three axes be the normal to the surface, the tangent to the curve, and a perpendicular to both at the same point, which will be in the plane tangent to the surface. The forces R and $\frac{v^2}{\rho}$, resolved according to the normal, constitute the pressure already considered. In the tangent to the curve the force $\frac{v^2}{\rho}$ has no effect; and the force R, resolved in that direction, exerts no pressure, the effect of this resolved force being altogether expended in changing the velocity. The sum of the two forces reduced to the third axis is cipher: otherwise, the pressure would not be perpendicular to the surface. Accordingly, putting γ, γ' for the angles made by the directions of the forces R and $\frac{v^2}{\rho}$, with the third axis, this condition is expressed by the equation

$$\text{R}.\cos.\gamma+\frac{v^2}{\rho}.\cos.\gamma'=0. \qquad (c)$$

This equation will serve to determine the inclination of the plane of the osculating circle to the constraining surface at each point. For the tangent to the trajectory is the intersection of the plane of the osculating circle and the tangent plane: and the angle γ' being contained by two lines in those planes, and perpendicular to their intersection, is that by which the inclination of those planes is measured.

When $\text{R}=0$, the last equation gives $\cos.\gamma'=0$, or $\gamma'=90°$. Which shows, that when the body is set in motion by an impulse, and not afterwards solicited by any moving force, the plane of the osculating circle is, at every point, perpendicular to the constraining surface.

The same thing would appear by supposing the constraining surface to be polygonal. In passing from one face to another, the body is deflected by a force perpendicular to that on which it enters: and as its motion on this face is compounded of that in the preceding, and of that due to the reaction, it follows, that the two contiguous elements of its path are in a plane perpendicular to the face containing the second of those elements. The same thing is to be understood of every two contiguous elements of the path described on the polygonal surface. And the observation being extended to the limiting curved surface, it is inferred that the radius of the circle of curvature of the trajectory is every where perpendicular to the surface.

This being the exclusive property of the shortest line that can be drawn on the surface, between two given points, it follows, that if a body moves on a curved surface, unsolicited by any accelerating or retarding force, the course it takes between any two points of its path, is that of the shortest line that can be drawn on the surface between those points. But this inference is not limited to the case of

R$=0$. For if this were a tangential force, such as the force of friction, or the resistance of a fluid medium, the angle γ would be right, and $\cos.\gamma=0$. In which case, equation (c) equally gives $\cos.\gamma'=0$. or, $\gamma'=90^{\circ}$.

It is scarcely requisite to observe, that when R$=0$. in which case the velocity is constant, the pressure is every where reciprocally proportional to the radius of curvature.

4. To exemplify this theory, let a body influenced by the force of gravity receive an impulse communicating to it the velocity v', and let the motion be restrained by a surface of revolution whose axis is vertical.

Putting r for the radius of the circle described by any point of the generating curve, and z for the abscissa; the equation of that curve will be of the form $pdz+qdr=0$. The normal at any point of the surface being in the plane of the generating curve passing through that point, the reaction of the surface may be resolved into two, of which one is in the vertical, and the other in the direction of r, or perpendicular to the axis of revolution. Putting s' for the arc of the generating curve, measured from the vertex, these resolved forces are $\text{N}.\frac{dr}{ds'}$. and $-\text{N}.\frac{dz}{ds'}$. Of which the latter is again resolved into $-\text{N}.\frac{dz}{ds'}.\frac{x}{r}$. $-\text{N}.\frac{dz}{ds'}.\frac{y}{r}$. in the directions of x and y respectively. Accordingly, the fundamental equations are

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-\text{N}.\frac{dz}{ds'}.\frac{x}{r}.\\ \frac{d^2y}{dt^2}&=-\text{N}.\frac{dz}{ds'}.\frac{y}{r}.\\ \frac{d^2z}{dt^2}&=-g+\text{N}.\frac{dr}{ds'}.\end{aligned}\right\}\qquad(d)$$

Eliminating N from the two first of these equations, we

get $\frac{y.d^2x - x.d^2y}{dt} = 0$. and by integration, $y.dx - x.dy = c.dt$. or putting ω for the angle contained between r and the axis of x,

$$r^2 d\omega = c dt. \qquad (e)$$

This equation shows that the areas projected on a horizontal plane are proportional to the times.

Equation (a) becomes in this case,

$$\frac{dx.d^2x + dy.d^2y + dz.d^2z}{dt^2} = -g dz.$$

or,

$$\frac{d(dx^2 + dy^2 + dz^2)}{dt^2} = -2g.dz.$$

which, by integration, gives

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = -2gz + c. \qquad \text{or, } v^2 = -2gz + c.$$

Putting z' for the height of the point of projection, the value of c will be expressed by the equation $c = v'^2 + 2gz'$. Wherefore, $v^2 = v'^2 + 2g(z' - z)$. Or putting h' for the height due to the velocity v', it is

$$v^2 = 2g(h' + z' - z). \qquad (f)$$

The path of the body shall be ascertained by finding an equation between ω and z. And for this, we have $\frac{dx^2 + dy^2 + dz^2}{dt^2} = 2g(h' + z' - z)$. and $cdt = r^2.d\omega$. between which, dt being eliminated, there is

$$c^2(dx^2 + dy^2 + dz^2) = 2r^4 g(h' + z' - z) d\omega^2.$$

Moreover, $dx^2 + dy^2$ is the square of the element of the arc described by the moving point, when projected on the plane of xy. Its value, therefore, is $dr^2 + r^2 d\omega^2$. Also, $r = fz$, given by the equation of the generating curve; and $dz = pdr$: wherein, p denotes the tangent of the inclination of the ele-

ment of the generating curve to the horizon. Making these substitutions, the equation is

$$c^2\left[\frac{1+p^2}{p^2}.dz^2+(fz)^2.d\omega^2\right]=2g(fz)^4(h'+z'-z)d\omega^2.$$

Whence,

$$d.\omega=\frac{c\sqrt{1+p^2}.dz}{p.fz\sqrt{2g(fz)^2.(h'+z'-z)-c^2}}. \qquad (g)$$

and

$$dt=\frac{(fz)^2.d\omega}{c}=\frac{\sqrt{1+p^2},fz.dz}{p\sqrt{2g(fz)^2(h'+z'-z)-c^2}}. \qquad (h)$$

Expanding the second member of this equation, and integrating any number of the terms, there is found an approximate value of t, as a function of z.

If the apsides are sought, we have only to make $\frac{dz}{dt}=0$. or $2g(fz)^2(h'+z'-'z)-c^2=0$. and to solve this equation for z.

To derive the expression of the pressure from the equations (d), the first of those equations is to be multiplied by $\frac{-x.dz}{rds'}$, the second by $\frac{-y.dz}{r.ds'}$, and the third by $\frac{dr}{ds'}$, and then adding, we have

$$\text{N}=\frac{r.dr.d^2z-dz(x.d^2x+y.d^2y)}{r.ds'.dt^2}+g.\frac{dr}{ds'}.$$

And as $dt=\frac{ds}{v}$, the equation is

$$\text{N}=\frac{v^2}{ds^2}\,\frac{[r.dr.d^2z-dz(x.d^2x+y.d^2y)]}{r.ds'}+g.\frac{dr}{ds'}.$$

To perceive the agreement of this formula, with the rule already given in Art. 2., it is to be observed, that $x^2+y^2=r^2$, twice differenced, gives $x.d^2x+y.d^2y=r.d^2r+dr^2-dx^2-dy^2$, which is reduced by the equations dz^2+dr^2

$=ds'^3$, $dx^2+dy^2+dz^2=ds^2$ to $x.d^2x+yd^2y=r.d^2r+ds'^2-ds^2$. Wherefore,

$$\text{N}=\frac{v^2}{ds^2}\,\frac{[r(dr.d^2z-dz.d^2r)+dz(ds^2-ds'^2)]}{r.ds'}+g.\frac{dr}{ds'}.$$

Further, ρ' being put for the radius of curvature of the generating curve at any point, its value is given by the equation $\rho'=\frac{ds'^3}{dr.d^2z-dz.d^2r}$. Wherefore,

$$\text{N}=\frac{v^2}{\rho'}.\frac{ds'^2}{ds^2}+v^2\frac{(ds^2-ds'^2)}{ds^2}\,\frac{dz}{r.ds'}+g.\frac{dr}{ds'}.$$

In this expression, we find the velocity resolved into two, viz. $v.\frac{ds'}{ds}$, according to the tangent of the generating curve; and $v.\frac{\sqrt{ds^2-ds'^2}}{ds}$, according to the tangent of the circle described by a point in that curve. The square of the former divided by ρ' is the centrifugal force arising from that velocity: and this force is directed in the normal. $\frac{v^2}{r}\frac{(ds^2-ds'^2)}{ds^2}$ is the centrifugal force, arising from the velocity reduced to the tangent of the circle. The direction of this is that of r, the radius of that circle; and being multiplied by $\frac{dz}{ds'}$, it is reduced to the normal. Wherefore, the sum of the two first terms in the value of N is the expression of the centrifugal force reduced to the normal. The last term is the force of gravity reduced to the same direction. Whence it appears, that the whole expression is equivalent to $\text{R}.\cos.\alpha+\frac{v^2}{\rho}.\cos.\alpha'$.

5. If it is required to find the velocity of projection, in order that the motion of the body should be horizontal, it is

to be observed that r and z are then to be treated as constant quantities. This, however, will not affect the differentials in the second members of the equations (d) which are there employed only hypothetically to denote the angles. This being observed, the third of those equations becomes $0=-g+\text{N}.\frac{dr}{ds'}$. Eliminating N between this and the first of those equations, we have $\frac{d^2x}{dt^2}=-g.\frac{x}{r}.\frac{dz}{dr}=-p.g.\frac{x}{r}$. But $x=r.\cos.\omega$. And this twice differenced for r constant, gives $\frac{d^2x}{dt^2}=-\frac{r.\cos.\omega.d\omega^2}{dt^2}$. Equating these values of $\frac{d^2x}{dt^2}$, we have $r.\frac{d\omega^2}{dt^2}=pg$. And as $\frac{rd\omega}{dt}=v$, the equation is $v^2=p.g.r$. Or putting for v^2 its value, viz. $2gh$, it is

$$h=\frac{pr}{2}.$$

Now p is the tangent of the angle made by the element of the generating curve with the horizontal ordinate r; wherefore, pr is the subtangent, *i. e.* the portion of the axis between the ordinate r and the tangent: and therefore, h, the height due to the velocity of projection, is half the subtangent. If the body is projected with this velocity in the tangent of a horizontal section of a surface of revolution, whose axis is vertical, it shall continue to move in that section.

The time of the revolution, or the periodic time of a body moving in this section, is $\text{T}=\frac{2\pi.r}{v}=\frac{2\pi r}{\sqrt{pgr}}=\frac{2\pi}{\sqrt{g}}.\sqrt{\frac{r}{p}}$. But $\frac{r}{p}$, is the subnormal, *i. e.* the portion of the axis contained between the normal and the horizontal ordinate r. Wherefore putting n for this line, we have

$$T = \frac{2\pi}{\sqrt{g}} . \sqrt{n}.$$

If the surface is that of a paraboloid, the subtangent is bisected at the vertex; wherefore, the velocity for circular motion is that due to the abscissa, or the portion of the axis cut off by the plane of the circle.

If the surface is that of a sphere, the velocity of projection for circular motion near the vertex or lowest point of the segment, as in the case of the paraboloid, is that due to the abscissa or height above that point. But this height increasing, the subtangent increases much more rapidly, becoming infinite for a height equal to the radius of the sphere. Wherefore for motion in a horizontal plane passing through the centre of the sphere, the velocity of projection should be infinite.

The time of the revolution given by the equation $T = \frac{2\pi}{\sqrt{g}} . \sqrt{n}$, varies as the square root of the portion of the vertical diameter between the centre of the sphere, and the plane of the circle described by the movement.

The condition of a body moving on the surface of a sphere, is the same as if it were suspended from the centre by a cord equal to the radius: the tension of the cord then taking the place of the pressure, and the reaction of the fixed point that of the constraining surface. Such is called a conical pendulum. And as the time of oscillation in a cycloid or small circular arc, was given by the equation $T = \pi \sqrt{\frac{l}{g}}$. it is evident that a conical pendulum shall perform its circular revolution in a time equal to that of the double oscillation of a pendulum whose length is equal to n, or the distance of the plane of the circle, described by the conical pendulum, from the point of suspension.

In the case of horizontal movement on a surface of revolution, the pressure on the surface is at once given by the third of the equations (d); which for $\frac{d^2z}{dt^2}=0$. is $N=g.\frac{ds'}{dr}$. The coefficient of g, in this equation, is the secant of the angle made by the element of the generating curve with the horizon; and this angle is equal to that between the normal and vertical axis. Wherefore, in the case of the horizontal movement of a body influenced by gravity, the pressure on the surface is the weight multiplied by the secant of the angle contained between the normal and axis.

It seemed proper to show how these conclusions respecting the horizontal motion of a heavy body on a surface of revolution described round a vertical axis, were to be obtained from the general theory of constrained motion. But the problems respecting such motions are readily solved, by equating the force of gravity and the centrifugal force, both reduced to the direction of the tangent to the generating curve: or which is the same thing, by resolving the force of gravity into two forces, of which, one is in the direction of the normal, and the other in that of the radius of the circle; and equating this last to the centrifugal force.

Putting n' for the normal, and n, as before, for the subnormal, the force in the direction of the former is $\frac{n}{n'}.g$. This which is the pressure on the surface, is equilibrated by the reaction. The central force is $\frac{r}{n}.g$. And this is equal and opposite to the centrifugal force. The expression for this force, already given, is $\frac{v^2}{r}$. or $\frac{2gh}{r}$. Wherefore, $\frac{2gh}{r}=\frac{r}{n}.g$.

or $h=\frac{r^2}{2n}$. *i. e.* the height due to the velocity of projection is equal to half the subtangent; which is the conclusion before obtained, respecting the velocity of projection required for the horizontal motion of a heavy body, on a surface of revolution about a vertical axis.

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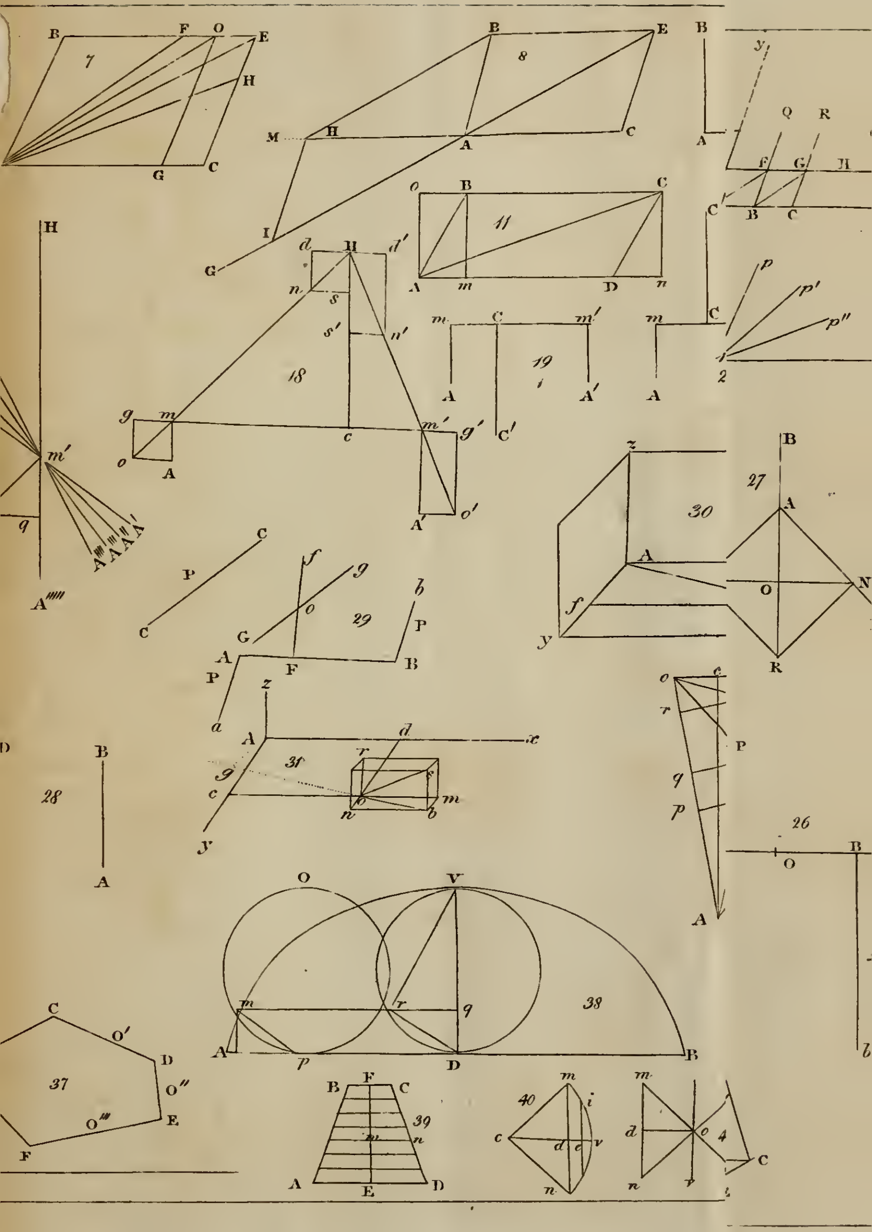
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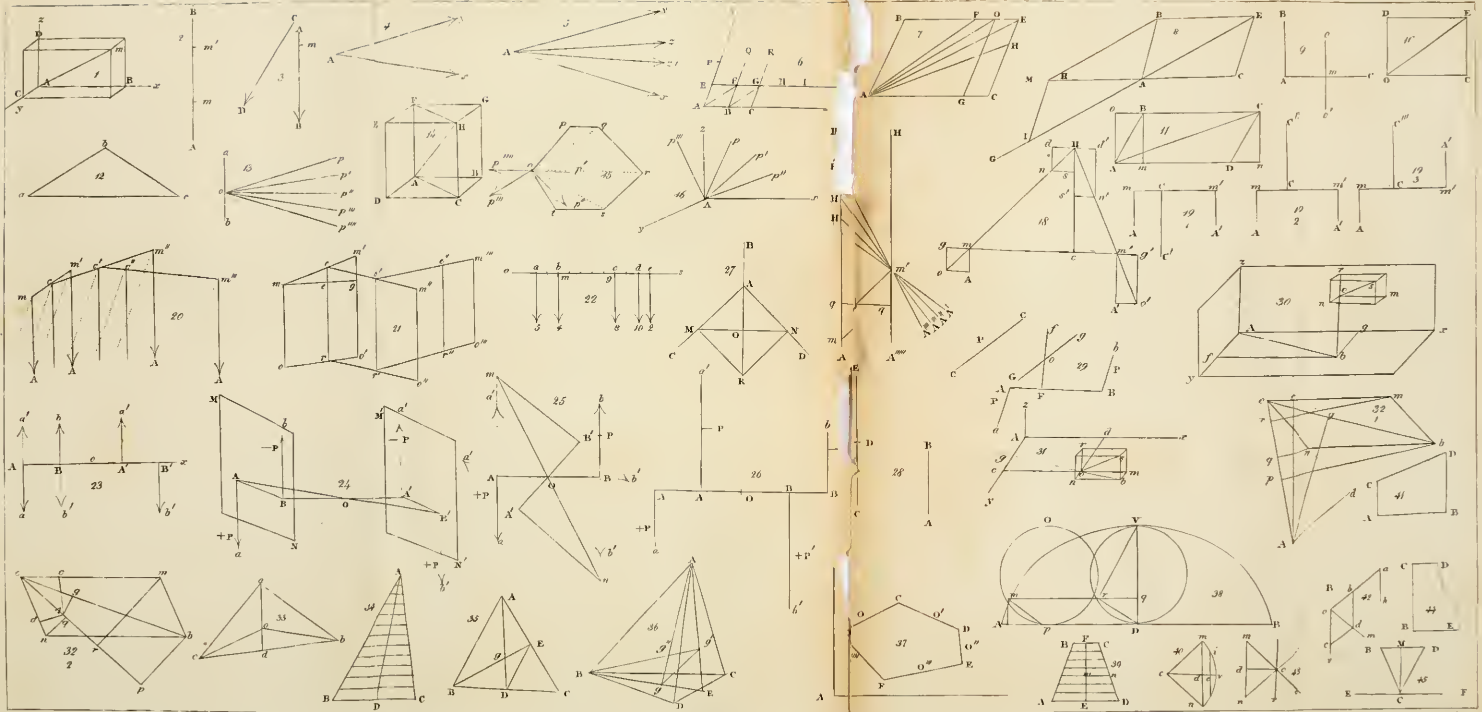

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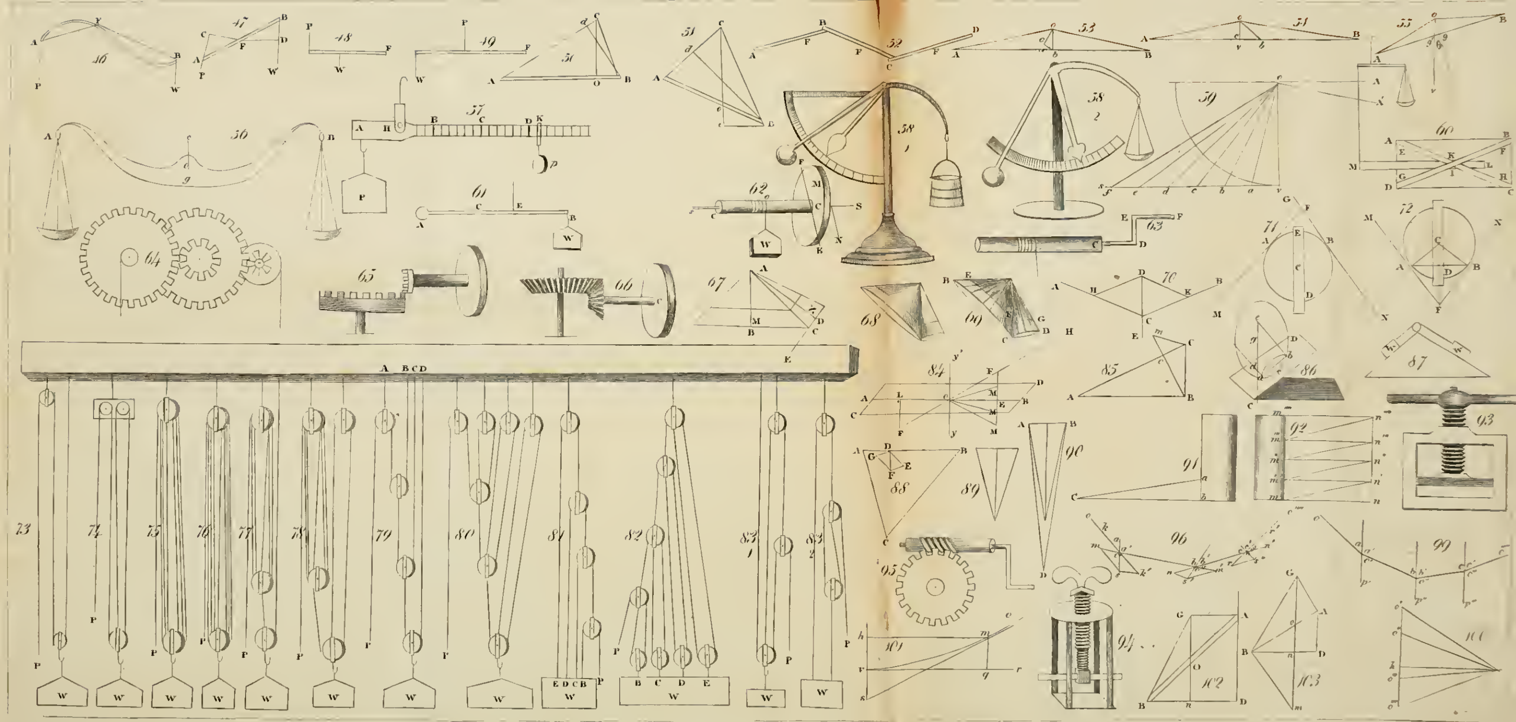
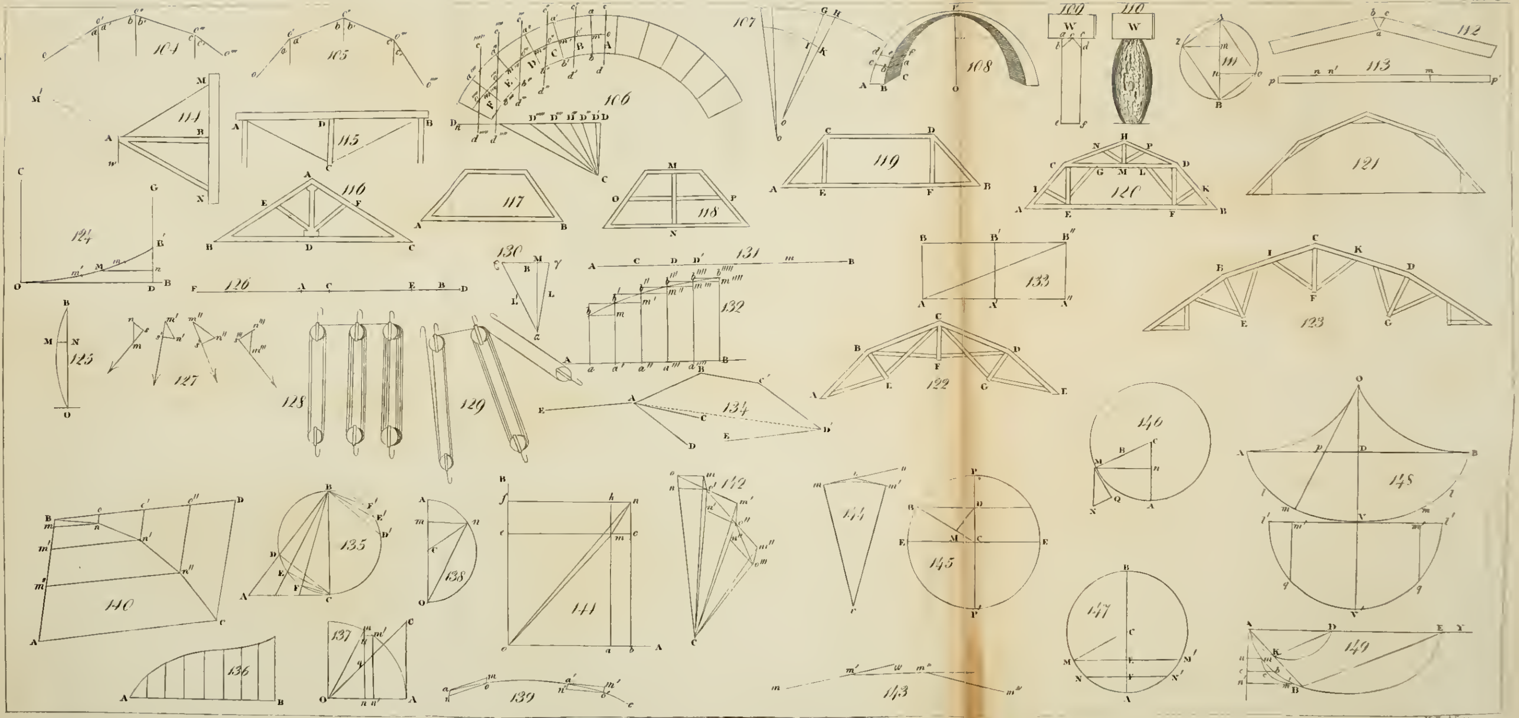
Plate 2

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